PAPER

SPORT: An Algorithm for Divisible Load Scheduling with Result Collection on Heterogeneous Systems

Abhay GHATPANDE[†a)], ***Student Member***, **Hidenori NAKAZATO**[†], ***Member***, **Olivier BEAUMONT**[††], ***Nonmember***, ***and*** **Hiroshi WATANABE**[†], ***Member***

SUMMARY Divisible Load Theory (DLT) is an established mathematical framework to study Divisible Load Scheduling (DLS). However, traditional DLT does not address the scheduling of results back to source (i.e., *result collection*), nor does it comprehensively deal with *system heterogeneity*. In this paper, the DLSRCHETS (DLS with Result Collection on HETerogeneous Systems) problem is addressed. The few papers to date that have dealt with DLSRCHETS, proposed simplistic LIFO (Last In, First Out) and FIFO (First In, First Out) type of schedules as solutions to DLSRCHETS. In this paper, a new polynomial time heuristic algorithm, SPORT (System Parameters based Optimized Result Transfer), is proposed as a solution to the DLSRCHETS problem. With the help of simulations, it is proved that the performance of SPORT is significantly better than existing algorithms. The other major contributions of this paper include, for the first time ever, (a) the derivation of the condition to identify the presence of idle time in a FIFO schedule for two processors, (b) the identification of the limiting condition for the optimality of FIFO and LIFO schedules for two processors, and (c) the introduction of the concept of *equivalent processor* in DLS for heterogeneous systems with result collection.

key words: *divisible load scheduling, heterogeneous systems, result collection*

1. Introduction

Divisible loads form a special class of parallelizable applications, which if given a large enough volume, can be *arbitrarily* partitioned into any number of independently- and identically-processable *load fractions*. Examples of applications that satisfy this divisibility property include massive data-set processing, image processing, signal processing, computation of Hough transforms, database search, simulations, and matrix computations. Divisible Load Theory (DLT) is the mathematical framework that has been established to study Divisible Load Scheduling (DLS) [1]–[22].

DLT has gained popularity because of its simplicity and deterministic nature. In a star connected network where the center of the star acts as the master and holds the entire load to be distributed, and the points of the star form the set of slave processors, the basic principle of DLT to determine an optimal schedule is the AFS (All nodes Finish Simultaneously) policy [17]. This states that the optimum schedule is one in which the load is distributed such that all the nodes involved in the computation finish processing their individual load fractions at the same time.

Manuscript received October 30, 2007.
Manuscript revised March 17, 2008.

[†]The authors are with Waseda University, Tokyo, 169-0051 Japan.
[††]The author is with INRIA Futurs — LaBRI, France.
a) E-mail: abhay@toki.waseda.jp

DOI: 10.1093/ietcom/e91–b.8.2571

In the AFS policy, after the nodes finish computing their individual load fractions, no results are returned to the source. In most practical applications this is an unrealistic assumption, and the result collection phase contributes significantly to the total execution time, unless each node returns a floating point or boolean value. This is not that uncommon, but the other cases need due consideration. All papers that have addressed result collection to date, have advocated simplistic LIFO (Last In, First Out) and FIFO (First In, First Out) sequences or variants thereof as solutions [1], [16], [17], [23]–[26]. It has been proved in [23] that LIFO and FIFO are not always optimal, and as this paper shows, the performance of these algorithms varies widely depending on whether it is the network links that are heterogeneous, or the processor speeds, or both.

Several papers have dealt with DLS on heterogeneous systems to date [1], [4], [23]–[26]. As far as can be judged, no paper has given a satisfactory solution to the scheduling problem where both the network bandwidth and computation capacities of the nodes are different, and the result transfer to the source is explicitly considered, i.e., to the DLS with Result Collection on HETerogeneous Systems (DLSRCHETS) problem. Along with the AFS policy, there are two assumptions that have implicitly pervaded DLT literature to date: (a) load is allocated to *all* processors, and (b) processors are never idle. The presence of idle time in the optimal schedule, which is a very important issue, has been overlooked in DLT work on result collection and heterogeneity. For the first time, [24], [25] proved that the optimal FIFO schedule can have a single processor with idle time, and that this processor can always be chosen to be the one to which load is allocated last.

In this paper, the completely general form of DLSRCHETS is tackled, with no assumptions being made regarding the number of processors allocated load, the network and node heterogeneity, or on the presence (or absence) of idle time. The main contribution of this paper is the establishment of the theoretical base for the SPORT (System Parameters based Optimized Result Transfer) algorithm, and the proof of its performance through simulation. This paper includes for the first time ever, (a) the derivation of the condition to identify the presence of idle time in a FIFO schedule for two processors, (b) the identification of the limiting condition for the optimality of FIFO and LIFO schedules for two processors, and (c) the introduction of the concept of an *equivalent processor* in DLS for heterogeneous systems with result collection.

SPORT does not necessarily use all processors and determines the number of processors to be used based on the system parameters (computation and communication capacities). SPORT simultaneously finds the sequence of load allocation and result collection, and the load distribution (i.e., the load fractions to be allocated to the processors). Given processors sorted in the order of decreasing network link bandwidth, the complexity of SPORT is, where m is the number of available processors $O(m)$.

The rest of the paper is as follows. In Sect. 2, the system model and the DLSRCHETS problem is described. The basic two processor system, the manner of evaluation of its optimum schedule, and the process of combination of the processors into an *equivalent processor* are given in Sect. 3. In Sect. 4, the SPORT algorithm is proposed. Simulation results are presented in Sect. 5. Section 6 provides the conclusion and future work.

2. System Model and Problem Definition

The divisible load $\mathcal{J}$ is to be distributed and processed on a heterogeneous star network $\mathcal{H} = (\mathcal{P}, \mathcal{L}, \mathcal{E}, C)$ as shown in Fig. 1, where $\mathcal{P} = \{p_0, \ldots, p_m\}$ is the set of $m + 1$ processors, and $\mathcal{L} = \{l_1, \ldots, l_m\}$ is the set of m network links that connect the master scheduler (source) p_0 at the center of the star, to the slave processors $p_1, \ldots, p_m$. $\mathcal{E} = \{E_1, \ldots, E_m\}$ is the set of computation parameters of the slave processors, and $C = \{C_1, \ldots, C_m\}$ is the set of communication parameters of the network links. E_k is the reciprocal of the speed of processor p_k, and C_k is the reciprocal of the bandwidth of link l_k. Both are defined in time units per unit load, i.e., p_k takes E_k time units to process a unit load transmitted to it from p_0 in C_k time units over the link l_k, and $E_k, C_k \geq 0$, E_k, C_k not simultaneously zero, for $k \in \{1, \ldots, m\}$.

The values in $\mathcal{E}$ and C are assumed to be deterministic and available at the source. Based on these parameter values, the source p_0 splits $\mathcal{J}$ into parts (fractions) $\alpha_1, \ldots, \alpha_m$ and sends them to the respective processors $p_1, \ldots, p_m$ for computation. Each such set of m fractions is known as a *load distribution* $\alpha = \{\alpha_1, \ldots, \alpha_m\}$. The source does not retain any part of the load for computation. If it does, then it can be modeled as an additional slave processor with computation parameter E_0 and communication parameter $C_0 = 0$.

All processors follow a *single-port and no-overlap* communication model, implying that processors can communicate with only one other processor at a time, and communication and computation *cannot* occur simultaneously. The processors are continuously and exclusively available during the course of execution of the entire process and have sufficient buffer capacity to receive the entire load fraction in a single installment from the source. The time taken for computation and communication is a linearly increasing function of the size of data.

The execution of the divisible load on each processor comprises of three distinct phases — the allocation phase, where data is sent to the processor from the source, the computation phase, where the data is processed, and the result collection phase, where the processor sends the processed data back to the source. The computation phase begins only after the entire load fraction allocated to that processor is received from the source. Similarly, the result collection phase begins only after the entire load fraction has been processed, and is ready for transmission back to the source. This is known as a *block based* system model, since each phase forms a block on the time line (see Fig. 2). The source starts receiving results from the child processors only after the entire load is distributed to the child processors first.

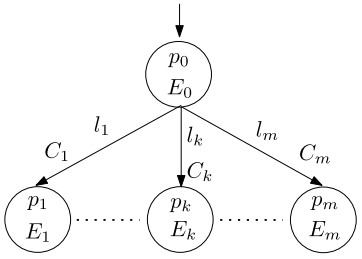


Fig. 1 Heterogeneous star network $\mathcal{H}$.

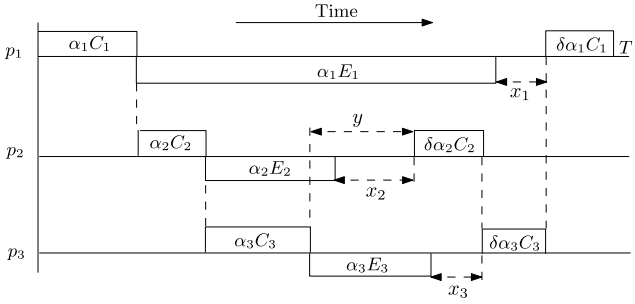


Fig. 2 A feasible schedule for $m = 3$.

For the divisible loads under consideration, such as image and video processing, Kalman filtering, matrix conversions, etc., the computation phase usually involves simple linear transformations, and the volume of returned results can be considered to be proportional to the amount of load received in the allocation phase. This is the accepted model for returned results in literature to date, [4], [17], [23]–[26]. If the allocated load fraction is α_k, then the returned result is equal to $\delta\alpha_k$, where $0 \leq \delta \leq 1$. The constant δ is application specific, and is the same for all processors for a particular load $\mathcal{J}$. For a load fraction α_k, $\alpha_k C_k$ is the transmission time from p_0 to p_k, $\alpha_k E_k$ is the time it takes p_k to perform the requisite processing on α_k, and $\delta\alpha_k C_k$ is the time it takes p_k to transmit the results back to p_0.

Let σ_a and σ_c be two permutations of order m that represent the allocation and collection sequences respectively, i.e., $\sigma_a[k]$ and $\sigma_c[k]$ denote the processor number that occurs at index $k \in \{1, \ldots, m\}$. Let $\sigma_a(l)$ and $\sigma_c(l)$ be two *lookup functions* that return the index of the processor $l \in \{1, \ldots, m\}$ in the allocation and collection sequences respectively. The DLSRCHETS problem is defined as follows:

DLSRCHETS (DLS WITH RESULT COLLECTION ON HETEROGE-

NEOUS SYSTEMS)

Given a heterogeneous network $\mathcal{H} = (\mathcal{P}, \mathcal{L}, \mathcal{E}, \mathcal{C})$, and a divisible load $\mathcal{J}$, find the sequence pair (σ_a, σ_c), and load distribution $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ that

MINIMIZE $\zeta = 0\alpha_1 + \ldots + 0\alpha_m + T$,
SUBJECT TO:

$$\sum_{j=1}^{\sigma_a(k)} \alpha_{\sigma_a[j]} C_{\sigma_a[j]} + \alpha_k E_k + \sum_{j=\sigma_c(k)}^{m} \delta\alpha_{\sigma_c[j]} C_{\sigma_c[j]} \leq T \qquad k = 1, \ldots, m \tag{1}$$

$$\sum_{j=1}^{m} \alpha_{\sigma_a[j]} C_{\sigma_a[j]} + \sum_{j=1}^{m} \delta\alpha_{\sigma_c[j]} C_{\sigma_c[j]} \leq T \tag{2}$$

$$\sum_{j=1}^{m} \alpha_j = \mathcal{J} \tag{3}$$

$$T \geq 0, \quad \alpha_k \geq 0 \qquad k = 1, \ldots, m \tag{4}$$

In the above formulation, for a pair (σ_a, σ_c), (1) imposes the no-overlap constraint. The single-port communication model is enforced by (2). The fact that the entire load is distributed amongst the processors is ensured by (3). This is known as the *normalization equation*. The non-negativity of the decision variables is ensured by constraint (4).

A linear programming problem in this form for a pair (σ_a, σ_c) can be solved using standard linear programming techniques in polynomial time [27]. There are $m!$ possible permutations each of σ_a and σ_c, and the linear program has to be evaluated $(m!)^2$ times to determine the globally optimal solution $(\sigma_a^*, \sigma_c^*, \alpha^*)$. Clearly, this is impractical to do for more than a few processors.

3. Analysis of Two Slave Processors

As mentioned in Sect. 1, traditional DLT assumes that load is always distributed to all the processors in a network, and that a processor is idle only up to the point where it starts receiving its load fraction. In the case of a heterogeneous network with result collection, this may not always be true. As shown in Fig. 2, idle time x_i may potentially be present in each processor i because it has to wait for another processor to release the communication medium for result transfer. In [24], [25] it has been proved that in the optimal solution to DLSRCHETS, $\forall i \in \{1 \ldots m\}$, $x_i = 0$, if and only if $y > 0$, and that there exists a unique $x_i > 0$ if and only if $y = 0$, where y is the intervening time interval between the end of allocation phase of processor $\sigma_a[m]$ and the start of result collection from processor $\sigma_c[1]$. For the FIFO schedule in particular, processor $\sigma_a[m]$ can always be selected to have idle time when $y = 0$, i.e., in the FIFO schedule, $x_{\sigma_a[m]} > 0$ if and only if $y = 0$. In the LIFO schedule, since $y > 0$ always, no processor has idle time, i.e., $\forall i \in \{1 \ldots m\}$, $x_i = 0$ always [23].

In the general case considered in this paper, for a pair (σ_a, σ_c), the solution to the linear program defined by (1) to (4) is completely determined by the values of δ, $\mathcal{E}$, $\mathcal{C}$, and it is not possible at this stage to predict which processor is the one that has idle time in the optimal solution. In fact, it is possible that not all processors are allocated load in the optimal solution (in which case some processors are idle throughout). The processors that are allocated load for computation are known as the *participating processors* (or *participants*).

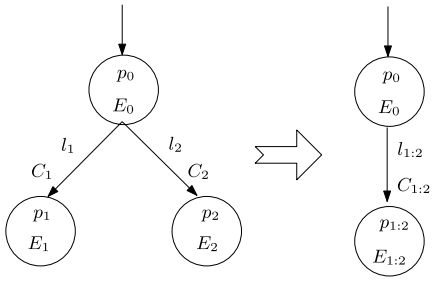


Fig. 3 Two-slave network and its equivalent network.

Thus any heuristic algorithm for DLSRCHETS must find both — the number of participants, and the load fractions allocated to them. A polynomial time heuristic algorithm, SPORT is proposed in this paper that does this simultaneously. The foundation of the SPORT algorithm is laid first by analyzing the case of a network with two slave processors.

3.1 Optimal Schedule in Two-Slave Network

The heterogeneous network $\mathcal{H} = (\mathcal{P}, \mathcal{L}, \mathcal{E}, \mathcal{C})$ with $m = 2$ is shown in Fig. 3. It is defined as in Sect. 2 by $\mathcal{P} = \{p_0, p_1, p_2\}$, $\mathcal{L} = \{l_1, l_2\}$, $\mathcal{E} = \{E_1, E_2\}$, and $\mathcal{C} = \{C_1, C_2\}$. Without loss of generality, it is assumed that $\mathcal{J} = 1$ and $C_1 \leq C_2$. No assumptions are possible regarding the relationship between E_1 and E_2, or $C_1 + E_1 + \delta C_1$ and $C_2 + E_2 + \delta C_2$ (or equivalently $C_1 + E_1$ and $C_2 + E_2$).

An important parameter, ρ_k, known as the *network parameter* is introduced, which shows how fast (or slow) the processor computation parameter E_k is with respect to the communication parameter C_k of its network link:

$$\rho_k = \frac{E_k}{C_k} \qquad k = 1, \ldots, m$$

The master p_0 distributes the load $\mathcal{J}$ between the two slave processors p_1 and p_2 so as to minimize the processing time T. Depending on the values of δ, $\mathcal{E}$ and $\mathcal{C}$, there are several possibilities:

1. Load is distributed to p_1 only with processing time
$$T^1 = C_1 + E_1 + \delta C_1 = C_1(1 + \delta + \rho_1) \tag{5}$$
2. Load is distributed to p_2 only with processing time
$$T^2 = C_2 + E_2 + \delta C_2 = C_2(1 + \delta + \rho_2) \tag{6}$$
3. Load is distributed to both p_1 and p_2. From [23]–[25], it can be proved that as long as $C_1 \leq C_2$, only the schedules shown in Figs. 4(a), 4(b), and 4(c) can be optimal

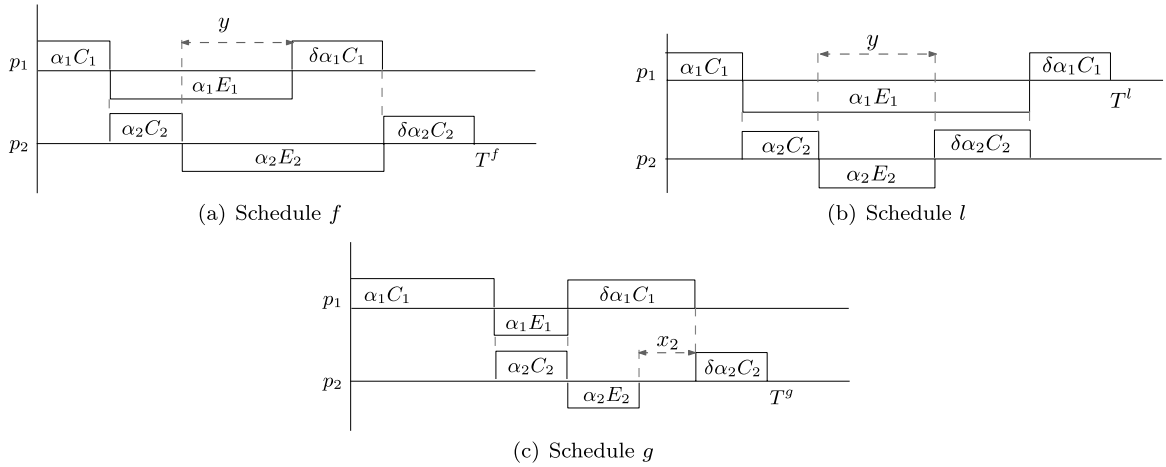


Fig. 4 Relevant schedules for network with $m = 2$.

for a two-slave network. These schedules are referred to as Schedule f, Schedule l, and Schedule g respectively. Superscripts f, l, and g are used to distinguish the three schedules. The processing times are given in Appendix.

A few lemmas to determine the optimal schedule for a two-slave network are now stated.

Lemma 1: It is always advantageous to distribute the load to both the processors, rather than execute it on the individual processors (for the system model under consideration).

Proof. From (A·4), (A·5), (A·16), (A·17), (A·24), and (A·25), it can be concluded that:

1. $\delta C_2 \le C_1(1+\delta+\rho_1) \Rightarrow T^f \le T^1$
2. $C_1 \le C_2 \Rightarrow T^f \le T^2$
3. $E_1 \ge 0 \Rightarrow T^l \le T^1$
4. $C_1 \le C_2 \Rightarrow T^l \le T^2$
5. $\delta C_2 \le C_1(1+\delta+\rho_1) \Rightarrow T^g \le T^1$
6. $C_1 \le C_2 \Rightarrow T^g \le T^2$

By assumption, $C_1 \le C_2$. Hence, from points 2, 4, and 6 above, execution time of Schedules f, l, and g is always smaller than T^2.

By definition, $E_1 > 0$. From points 1, 3, and 5 above, as long as $\delta C_2 \le C_1(1+\delta+\rho_1)$, execution time of Schedules f, l, and g is less than T^1.

Finally, if $\delta C_2 > C_1(1+\delta+\rho_1)$, then T^f and T^g are greater than T^1, but since T^l is always less than T^1, load can be distributed in Schedule l to reduce the processing time.

Thus, it is always advantageous to distribute load to two processors instead of one under the system model under consideration. ■

Lemma 1 is important because it helps SPORT determine the number of participants in a general schedule. Details are given in Sect. 4.1.

From Lemma 1, if $\delta C_2 \le C_1(1+\delta+\rho_1)$, then any of the Schedules f, g, or l could be optimal. The limiting condition between Schedule f and Schedule g is stated in the following lemma.

Lemma 2: $\rho_1\rho_2 \le \delta$ is a necessary and sufficient condition to indicate the presence of idle time in the FIFO schedule (i.e. Schedule g).

Proof. If the values of δ, $\mathcal{E}$, and C, are such that they necessitate the presence of idle time in a FIFO schedule, then the schedule can be reduced to one similar to that shown in Fig. 4(c).

In that case, idle time in processor p_2 occurs *only when* $\alpha_2^g E_2 \le \delta\alpha_1^g C_1$. From (A·22), this condition reduces to $\rho_1\rho_2 \le \delta$. ■

The simplicity of the condition to detect the presence of idle time in the FIFO schedule is both pleasing and surprising, and as far as can be judged, has been derived for the first time ever. Further confirmation of this condition is obtained in Sect. 3.2.

The following theorem can now be stated.

Theorem 1 (Optimal Schedule Theorem): The optimal schedule for a two-slave network can be found as follows:

1. If $\delta C_2 > C_1(1+\delta+\rho_1)$, then Schedule l is optimal.
2. If $\delta C_2 \le C_1(1+\delta+\rho_1)$, and both (A·22) and (A·26) hold, then Schedule g is optimal. Else if (A·26) does not hold, then Schedule l is optimal.
3. If $\delta C_2 \le C_1(1+\delta+\rho_1)$, (A·22) does not hold, and condition (A·18) holds, then Schedule f is optimal. Else if (A·18) does not hold, then Schedule l is optimal.

Proof. The proof follows directly from Lemmas 1 and 2, and (A·18) and (A·26). ■

All the conditions use only the data provided in the definition of the problem, with the exception of (A·18) which requires the computation of T^f.

It can be argued that the optimal schedule can be determined by directly computing the values of T^f, T^l, and T^g using (A·3), (A·14), and (A·23) respectively. While this fact cannot be denied, it defeats the ultimate purpose of

this research, which is to identify relationships between the system parameters that influence the optimality of different schedules.

Another very interesting insight into the problem is provided by (A· 18). The value $C_1C_2/(C_2-C_1)$ forms a limiting condition between the optimality of Schedules f and l. It can be seen that as long as T^f is smaller than this value, it is also smaller than T^l.

As the network links become homogeneous, the difference $(C_2 - C_1)$ becomes small, and Schedule f is likely to be optimal because T^f would be less than the large value $C_1C_2/(C_2-C_1)$. Similarly, as the network links become heterogeneous, the value $C_1C_2/(C_2 - C_1)$ becomes small, and Schedule l would tend to be optimal because T^f can easily exceed this value.

Rosenberg reached a similar conclusion with the help of simulations [26]. However this condition is analytically derived for the first time in DLT literature.

Once the optimal schedule (i.e., σ_a^* and σ_c^*) is known, it is trivial to calculate the optimal load distribution α^* using the equations in Appendix.

The optimal solution to DLSRCHETS, $(\sigma_a^*, \sigma_c^*, \alpha^*)$, for a network with two slave processors is a function of the system parameters and the application under consideration, because of which, no particular sequence of allocation and collection can be defined *a priori* as the optimal sequence. The optimal solution can only be determined once all the parameters become known.

3.2 Equivalent Processor and Equivalent Network

To extend the above result to the general case with m slave processors, the concept of an *equivalent processor* is introduced. Consider the system in Fig. 3. The processors p_1 and p_2 are replaced by a single equivalent processor $p_{1:2}$ with computation parameter $E_{1:2}$, connected to the root by an equivalent link $l_{1:2}$ with communication parameter $C_{1:2}$. The resulting network is called the *equivalent network*. The values of the equivalent parameters for the three schedules are given below.

In Fig. 5(a), the top half shows Schedule f for the two processors in the original network, while the bottom half shows the corresponding schedule for the equivalent network. Similarly Figs. 5(b) and 5(c) show the equivalent networks for Schedules l and g respectively. If the initial load distribution is $\alpha = \{\alpha_1, \alpha_2\}$, and the processing time is T, then the equivalent network satisfies the following properties:

- The load processed by $p_{1:2}$ is $\alpha_{1:2} = \alpha_1 + \alpha_2 = 1$.
- The processing time is unchanged and equal to T.
- The time spent in load distribution and result collection is unchanged, i.e., $\alpha_{1:2}C_{1:2} = \alpha_1C_1 + \alpha_2C_2$ and $\delta\alpha_{1:2}C_{1:2} = \delta\alpha_1C_1 + \delta\alpha_2C_2$.
- The time spent in load computation is equal to the intervening time interval between the end of allocation phase and the start of result collection phase, i.e.,
  - For Schedule f, $\alpha_{1:2}E^f_{1:2} = \alpha_1E_1 - \alpha_2C_2 = \alpha_2E_2 - \delta\alpha_1C_1$.
  - For Schedule l, $\alpha_{1:2}E^l_{1:2} = \alpha_2E_2 = \alpha_1E_1 - \alpha_2C_2 - \delta\alpha_2C_2$.
  - For Schedule g, $\alpha_{1:2}E^g_{1:2} = 0$.

This leads to the following theorem:

Theorem 2 (Equivalent Processor Theorem): In a heterogeneous network $\mathcal{H}$ with $m = 2$, the two slave processors p_1 and p_2 can be replaced without affecting the processing time T, by a single (virtual) equivalent processor $p_{1:2}$ with equivalent parameters $C_{1:2}$ and $E_{1:2}$, such that $C_1 \le C_{1:2} \le C_2$ and $E_{1:2} \le E_1, E_2$.

Proof. From (A· 6), the processing time of Schedule f can be written as,

$$T^f = \alpha_{1:2}C^f_{1:2} + \alpha_{1:2}E^f_{1:2} + \delta\alpha_{1:2}C^f_{1:2}$$

where

$$\alpha_{1:2} = \alpha^f_1 + \alpha^f_2 = 1$$

$$C^f_{1:2} = \frac{C_1C_2(r^f_1 + r^f_2)}{C_1r^f_1 + C_2r^f_2} \tag{7}$$

$$E^f_{1:2} = \frac{C_1C_2(\rho_1\rho_2 - \delta)}{C_1r^f_1 + C_2r^f_2} \tag{8}$$

Similarly, from (A· 19), the processing time of Schedule l can be written as,

$$T^l = \alpha_{1:2}C^l_{1:2} + \alpha_{1:2}E^l_{1:2} + \delta\alpha_{1:2}C^l_{1:2}$$

where

$$\alpha_{1:2} = \alpha^l_1 + \alpha^l_2 = 1$$

$$C^l_{1:2} = \frac{C_1C_2(r^l_1 + r^l_2)}{C_1r^l_1 + C_2r^l_2} \tag{9}$$

$$E^l_{1:2} = \frac{C_1C_2\rho_1\rho_2}{C_1r^l_1 + C_2r^l_2} \tag{10}$$

From (A· 23), the processing time of Schedule g can be written as

$$T^g = \alpha_{1:2}C^g_{1:2} + \alpha_{1:2}E^g_{1:2} + \delta\alpha_{1:2}C^g_{1:2}$$

where,

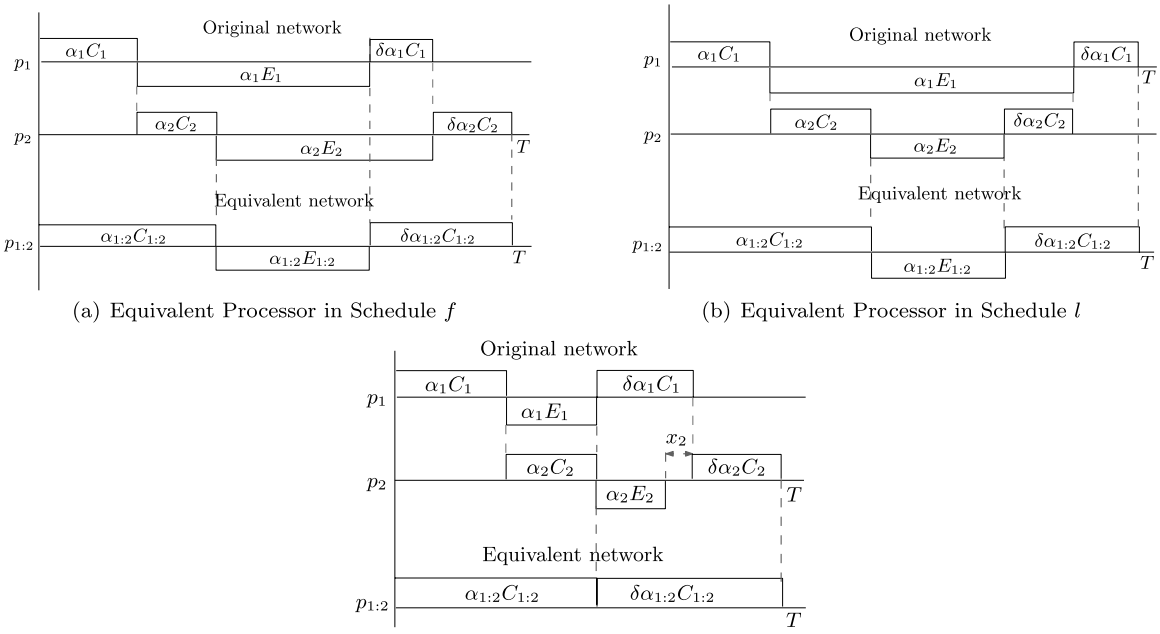
$$\alpha_{1:2} = \alpha^g_1 + \alpha^g_2 = 1$$

$$C^g_{1:2} = \frac{C_1C_2(1 + \rho_1)}{C_1\rho_1 + C_2} \tag{11}$$

$$E^g_{1:2} = 0 \tag{12}$$

It can be easily verified that these representations satisfy the properties of equivalent processor mentioned above.

For Schedule f, l, and g, $\alpha_{1:2}C_{1:2} = \alpha_1C_1 + \alpha_2C_2$, $\alpha_1 = 1 - \alpha_2$, and $\alpha_{1:2} = 1$, so the equivalent communication parameter can be written as $C_{1:2} = C_1 + \alpha_2(C_2 - C_1)$.



(a) Equivalent Processor in Schedule f

(b) Equivalent Processor in Schedule l

(c) Equivalent Processor in Schedule g

Fig. 5 Timing diagrams for equivalent processor.

Since by definition, $0 \le \alpha_2 \le 1$, it immediately follows that $C_1 \le C_{1:2} \le C_2$.

Similarly, from the definition of equivalent network, $E^f_{1:2} = \alpha_1 E_1 - \alpha_2 C_2 = \alpha_2 E_2 - \delta\alpha_1 C_1$ and $E^l_{1:2} = \alpha_2 E_2 = \alpha_1 E_1 - \alpha_2 C_2(1+\delta)$. Since $0 \le (\alpha_1,\ \alpha_2) \le 1$, it follows that $E^f_{1:2} \le E_1, E_2$ and $E^l_{1:2} \le E_1, E_2$.

Finally, $E^g_{1:2} = 0 \Rightarrow E^g_{1:2} \le E_1, E_2$, since by definition, $E_1, E_2 > 0$. ■

The equivalent processor for Schedule f provides additional confirmation of the condition for the presence of idle time in a FIFO schedule (i.e. use of Schedule g). It is known that idle time can exist in a FIFO schedule only when the intervening time interval $y = 0$. According to the definition of equivalent processor, this interval corresponds to the equivalent computation capacity $E^f_{1:2}$. From (8), this value becomes zero only when $\rho_1\rho_2 - \delta = 0$. Thus, if $\rho_1\rho_2 < \delta$, then idle time must exist in the FIFO schedule.

The equivalent processor enables replacement of two processors by a single processor with communication parameter with a value that lies between the values of communication parameters of the original two links. Because of this property, if the processors are arranged so that $C_1 \le C_2 \le \ldots \le C_m$, and two processors are combined at a time sequentially, then the resultant equivalent processor does not disturb the order of the sequence. This property is exploited in the SPORT algorithm, which is described next.

4. Proposed Algorithm

The proposed SPORT algorithm is as follows.

Algorithm 1 (SPORT):

1: Arrange $p_1, \ldots, p_m$ *s.t.* $C_1 \le C_2 \le \ldots \le C_m$
2: $\sigma_a \leftarrow 1, \sigma_c \leftarrow 1, \alpha_1 \leftarrow 1$
3: **for** $k := 2$ to m **do**
4: $C_1 \leftarrow C_{1:k-1}, E_1 \leftarrow E_{1:k-1}, C_2 \leftarrow C_k, E_2 \leftarrow E_k$
5: **if** $\delta C_2 > C_1(1+\delta+\rho_1)$ **then**
6: /* $T^l < T^f, T^g$, use Schedule l */
7: **call** `schedule_lifo`(C_1, C_2, E_1, E_2)
8: **else**
9: /* Need to check other conditions */
10: **if** $\rho_1\rho_2 \le \delta$ **then**
11: /* Possibility of idle time */
12: **if** $C_2 \le C_1\Big(1 + \dfrac{(1+\rho_1)\rho_2}{\delta(1+\delta+\rho_2)}\Big)$ **then**
13: /* $T^g < T^l$, use Schedule g */
14: **call** `schedule_idle`(C_1, C_2, E_1, E_2)
15: **break for**
16: **else**
17: /* $T^l < T^g$, use Schedule l */
18: **call** `schedule_lifo`(C_1, C_2, E_1, E_2)
19: **end if**
20: **else**
21: /* No idle time present */
22: **if** $T^f \le \dfrac{C_1 C_2}{C_2 - C_1}$ **then**
23: /* $T^f < T^l$, use Schedule f */
24: **call** `schedule_fifo`(C_1, C_2, E_1, E_2)
25: **else**
26: /* $T^l < T^f$, use Schedule l */
27: **call** `schedule_lifo`(C_1, C_2, E_1, E_2)
28: **end if**
29: **end if**
30: **end if**
31: **end for**

32: $n \leftarrow$ numberOfProcessorsUsed
33: /* Find load fractions */
34:
$$\alpha_k \leftarrow \begin{cases} \alpha_k \cdot \prod_{j=2}^{n} \alpha_{1:j} & \text{if } k = 1 \\ \alpha_k \cdot \prod_{j=k}^{n} \alpha_{1:j} & \text{if } k = 2, \ldots, n \end{cases}$$
35: $T \leftarrow C_{1:n} + E_{1:n} + \delta C_{1:n}$

procedure schedule_idle(C_1, C_2, E_1, E_2)

1: $\alpha_{1:k-1} \leftarrow \dfrac{C_2}{C_1\rho_1 + C_2}$
2: $\alpha_k \leftarrow \dfrac{C_1\rho_1}{C_1\rho_1 + C_2}$
3: /* Update sequences for FIFO */
4: $\sigma_a \leftarrow [\sigma_a\ k]$
5: $\sigma_c \leftarrow [\sigma_c\ k]$
6: /* Compute equivalent processor parameters */
7: $C_{1:k} \leftarrow \dfrac{C_1C_2(1+\rho_1)}{C_1\rho_1 + C_2}$
8: $E_{1:k} \leftarrow 0$
9: numberOfProcessorsUsed $\leftarrow k$
10: **return**

end procedure

procedure schedule_lifo(C_1, C_2, E_1, E_2)

1: $r_1^l \leftarrow \rho_1$
2: $r_2^l \leftarrow 1 + \delta + \rho_2$
3: $\alpha_{1:k-1} \leftarrow \dfrac{C_2 r_2^l}{C_1 r_1^l + C_2 r_2^l}$
4: $\alpha_k \leftarrow \dfrac{C_1 r_1^l}{C_1 r_1^l + C_2 r_2^l}$
5: /* Update sequences for LIFO */
6: $\sigma_a \leftarrow [\sigma_a\ k]$
7: $\sigma_c \leftarrow [k\ \sigma_c]$
8: /* Compute equivalent processor parameters */
9: $C_{1:k} \leftarrow \dfrac{C_1C_2(r_1^l + r_2^l)}{C_1 r_1^l + C_2 r_2^l}$
10: $E_{1:k} \leftarrow \dfrac{C_1C_2\rho_1\rho_2}{C_1 r_1^l + C_2 r_2^l}$
11: numberOfProcessorsUsed $\leftarrow k$
12: **return**

end procedure

procedure schedule_fifo(C_1, C_2, E_1, E_2)

1: $r_1^f \leftarrow \delta + \rho_1$
2: $r_2^f \leftarrow 1 + \rho_2$
3: $\alpha_{1:k-1} \leftarrow \dfrac{C_2 r_2^f}{C_1 r_1^f + C_2 r_2^f}$
4: $\alpha_k \leftarrow \dfrac{C_1 r_1^f}{C_1 r_1^f + C_2 r_2^f}$
5: /* Update sequences for FIFO */
6: $\sigma_a \leftarrow [\sigma_a\ k]$
7: $\sigma_c \leftarrow [\sigma_c\ k]$
8: /* Compute equivalent processor parameters */
9: $C_{1:k} \leftarrow \dfrac{C_1C_2(r_1^f + r_2^f)}{C_1 r_1^f + C_2 r_2^f}$
10: $E_{1:k} \leftarrow \dfrac{C_1C_2(\rho_1\rho_2 - \delta)}{C_1 r_1^f + C_2 r_2^f}$
11: numberOfProcessorsUsed $\leftarrow k$
12: **return**

end procedure

4.1 Algorithm Explanation

If $m = 2$, finding the optimal schedule and load distribution is trivial. In the following, it is assumed that $m \geq 3$. At the start, the processors are arranged so that $C_1 \leq C_2 \leq \ldots \leq C_m$, and two processors with the fastest communication links are selected. The optimal schedule and load distribution for the two processors are found according to Theorem 1. If Schedule f or l is found optimal, then the two processors are replaced by their equivalent processor. In either case, since $C_1 \leq C_{1:2} \leq C_2$, the ordering of the processors does not change. In the subsequent iteration, the equivalent processor and the processor with the next fastest communication link are selected and the steps are repeated until either all processors are used up, or Schedule g is found to be optimal. If Schedule g is found to be optimal in any iteration, then the algorithm exits after finding the load distribution for that iteration.

An example of how the SPORT algorithm works for a network with $m = 3$ is given in Fig. 6. In Fig. 6, Schedule l is found to be optimal for processors p_1 and p_2, and Schedule f is optimal for their equivalent processor $p_{1:2}$ and the processor p_3. The resulting timing diagrams are as shown.

Now assume that this network has several more processors. Since SPORT adds processors one by one to the set of participants, Lemma 1 implies that the addition should be *greedy*, i.e. as many processors as possible should be used to minimize the processing time. If in the third iteration,

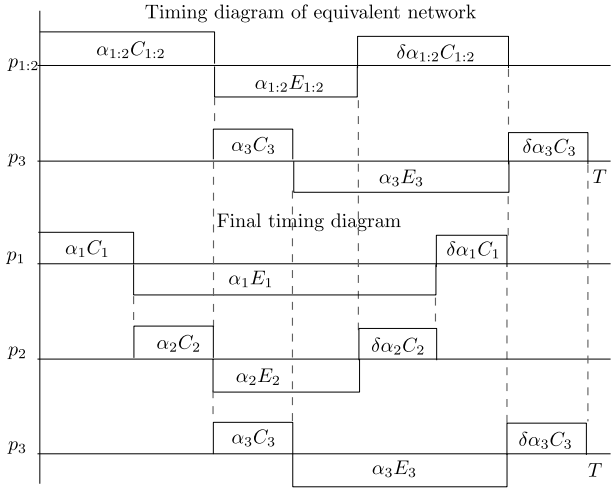


Fig. 6 An example of SPORT algorithm.

Schedule g is found to be optimal for the processors $p_{1:3}$ and p_4, then in that case, the intervening time interval between the end of load allocation to p_4 and the start of result collection from p_2 would be zero. Since additional processors are always inserted (allocated load) within this interval, it would not be possible to allocate load to any more processors, and the algorithm exits. Thus Schedule g forms the logical termination criterion for the algorithm unless all processors in the network are used. When Schedule g is found to be the optimal schedule, all remaining processors are allocated zero load, and their index in the allocation and collection sequence is assigned zero value.

The computation of the allocation and collection sequences is straightforward. The allocation sequence σ_a is maintained in the order of decreasing communication link bandwidth of the processors. Irrespective of the schedule found optimal in iteration k, k is always appended to σ_a. The collection sequence σ_c is constructed in the following manner:

- If Schedule f or g is found optimal in iteration k, k is appended to σ_c.
- If Schedule l is found optimal in iteration k, k is prepended to σ_c.

The calculation of load distribution to the processors occurs simultaneously with the search for the optimal schedule. As shown in Fig. 7, the algorithm creates a *binary tree* of load fractions. If the number of processors participating in the computation is n, $2 \leq n \leq m$, the root node of the binary tree is $\alpha_{1:n}$ and the leaf nodes represent the final load fractions allocated to the processors. The value of the root node need not be calculated as it is equal to one. The individual load fractions, α_k, are initially assigned value α'_k (say), and then updated at the end as:

$$\alpha_k = \begin{cases} \alpha'_k \cdot \prod_{j=2}^{n} \alpha_{1:j} & \text{if } k = 1 \\ \alpha'_k \cdot \prod_{j=k}^{n} \alpha_{1:j} & \text{if } k = 2, \ldots, n \end{cases} \tag{13}$$

This is equivalent to traversing the binary tree from the root to each leaf node and taking the product of the nodes encountered (see Fig. 7). This calculation can be easily implemented in $O(m)$ time by starting with the computation of α_n, and storing the values of the product terms (i.e. $\prod \alpha_{1:j}$) for each processor and then using that value for the next processor.

Once the sequences (σ_a, σ_c) and load distribution α are found, calculating the processing time is trivial. The processing time is simply the sum of the (equivalent) parameters of the equivalent processor $p_{1:n}$, i.e., $T = C_{1:n} + E_{1:n} + \delta C_{1:n}$.

4.2 Complexity and Discussion

In SPORT, defining the allocation sequence by sorting the values of C_k requires $O(m \log m)$ time, while finding the collection sequence and load distribution requires $O(m)$ time in the worst case. Thus, if sorted values of C_k are given, then the overall complexity of the algorithm is polynomial in m and is equal to $O(m)$.

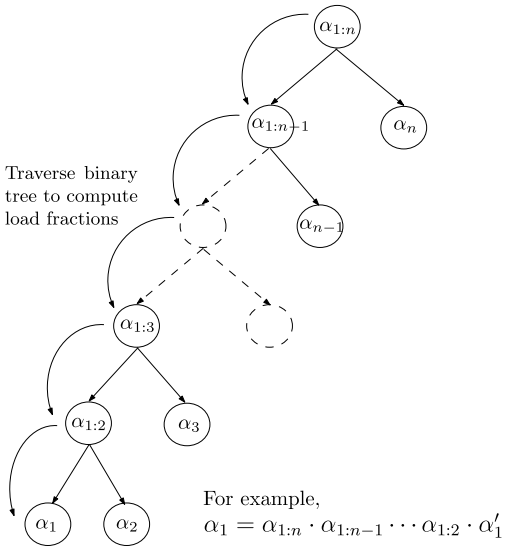


Fig. 7 Calculating the load fractions.

The equivalent processor method works because:

- The equivalent processor maintains the positions of the processors in the allocation sequence in order of decreasing communication bandwidth (increasing value of communication parameter).
- Two processors in each iteration are kept immediately successive in the final ordering.
- All equations are linear. So irrespective of the final time interval T, the respective positions and load fractions are not changed.

It follows that if latencies are considered (i.e. computation and communication costs are affine functions of the size of data) in the system model, then the equivalent processor method may not work. However most applications of DLS where result collection is important, involve transfers of large volumes of data, and latencies can be safely ignored.

As noted earlier in Sect. 3.1, when result collection phase is considered along with heterogeneous networks, at this stage at least, it is not possible to *a priori* define any single sequence as *the* optimum sequence for allocation or collection. Nor is it possible to determine a criterion for the optimal number of processors to be used in the computation. The number of processors used and the optimal sequences depend on the communication and computation parameters of the processors and the application under consideration (δ). Because of this, it is also not possible to derive closed-form equations for the load fractions or the total processing time.

SPORT *does not guarantee* a globally optimal solution to the DLSRCHETS problem. For example, if $\mathcal{C} = \{10, 20, 30\}$, $\mathcal{E} = \{5, 15, 25\}$, and $\delta = 0.5$, the globally optimal solution is $\sigma_a^* = \{1, 2, 3\}$, $\sigma_c^* = \{2, 3, 1\}$, and $\alpha^* = \{0.88, 0.08, 0.04\}$, while the solution found by SPORT is $\sigma_a = \{1, 2, 3\}$, $\sigma_c =$

$\{3, 2, 1\}$, and $\alpha = \{0.88, 0.1, 0.02\}$. However, since the solution (SPORT) is built on locally optimal values by considering two processors at a time, the error as compared to the global optimum is reduced to some degree.

Finding the conditions for the minimization of the error is a part of the future work.

5. Simulation Results and Analysis

In the simulations, all E_k and C_k are defined in the same time units. On open networks such as the Internet, and Grid computing platforms, it is not unusual for processors to have widely varying values of E_k and C_k, with the ratios $\min(E_k) : \max(E_k)$ and $\min(C_k) : \max(C_k)$ reaching 1:100 [28]. Further, they can appear in any combination. For example, a fast processor may have a very slow network connection, while a processor with a fast link may be overloaded and not have enough computation speed. Along with system heterogeneity, it is important to verify the effect of the application on the algorithms. To rigorously test the performance of SPORT, several simulations were performed with different ranges for E_k, C_k, and δ.

5.1 Simulation Set A

The performance of SPORT was compared to four algorithms, viz. OPT, FIFOC, LIFOC, and ITERLP, each of which is described below.

The globally optimal schedule OPT is obtained after evaluation of the linear program for all possible $(m!)^2$ permutations of (σ_a, σ_c). The MATLAB™ linear program solver `linprog` is used to determine the optimal solution to the linear program defined by constraints (1) to (4) for each permutation pair. The processing time for each pair is calculated, and the sequence pair and load distribution that results in the minimum processing time is selected as the OPT solution. This ensures that the minimal set of processors is used and the optimal processing time is found.

LIFOC and FIFOC heuristics are as follows. In FIFOC, processors are allocated load and result are collected in the order of decreasing communication link bandwidth of the processors. In LIFOC, load allocation is in the order of decreasing communication link bandwidth of the processors, while result collection is the reverse order of increasing communication link bandwidth of the processors.

For example let $\mathcal{C} = \{20, 10, 15\}$, $\mathcal{E} = \{5, 15, 10\}$. The processors are first sorted in the order of decreasing communication link bandwidth (i.e. by increasing value of C_k). The sorted processor numbers give the allocation sequence $\sigma_a = \{2, 3, 1\}$ for both FIFOC and LIFOC. For FIFOC, the result collection sequence, is the same as σ_a, i.e. $\sigma_c\big|_{\text{FIFOC}} = \{2, 3, 1\}$, and for LIFOC, the result collection sequence is the reverse of σ_a, i.e. $\sigma_c\big|_{\text{LIFOC}} = \{1, 3, 2\}$. For FIFOC, the sequence pair (σ_a, σ_c) so obtained, along with the sets $\mathcal{C}$ and $\mathcal{E}$, and δ, are used to construct the linear program defined by constraints (1) to (4), which is passed to the linear program solver `linprog`, that determines the optimal FIFOC solution. For LIFOC, using the transformation explained in [23], the optimal solution is found by using the closed form equations given in [14].

The ITERLP heuristic finds a solution by iteratively solving linear programs as follows. Processors are first sorted by increasing value of C_k (i.e., decreasing value of communication link bandwidth). The first two processors are selected and the optimal (σ_a, σ_c) pair (the one with the lowest processing time for the two processors) is determined by solving the linear program defined by the constraints (1) to (4) four times for each permutation of the σ_a and σ_c. The next processor in the sequence is added in the next iteration. The new processor can be interleaved at any position in (σ_a, σ_c), but with an additional constraint that the relative positions of processors already determined are maintained. By constraining the number of possible sequences in this manner, if the number of processors in an iteration is k, $k \leq m$, then k^2 linear programs are solved in that iteration instead of the possible $(k!)^2$.

For example, if the optimal sequences at the end of the first iteration are $\sigma_a^1 = \{1, 2\}$ and $\sigma_c^1 = \{2, 1\}$, then in the second iteration, the set of possible allocation sequences is $\Sigma_a^2 = \{(3, 1, 2), (1, 3, 2), (1, 2, 3)\}$, and the set of possible collection sequences is $\Sigma_c^2 = \{(3, 2, 1), (2, 3, 1), (2, 1, 3)\}$. In any iteration, if processor k is allocated zero load, then the algorithm terminates and does not proceed to the next iteration with $k + 1$ processors. This is similar to SPORT, except for the fact that equivalent processor cannot be used, and in the worst case, $\sum_{k=1}^{m} k^2 = O(m^3)$ linear programs have to be solved. It is computationally much too expensive to be used practically for large values of m, but can be used to compare the performance of other heuristic algorithms because of its near-optimal performance.

Preliminary simulations for other heuristic algorithms, viz. FIFO, LIFO, FIFOE, LIFOE, and SUMCE, revealed large errors in favor of SPORT, and it was decided not to pursue them further. The solutions to FIFO and LIFO are calculated similar to FIFOC and LIFOC except for the fact that the processors are not initially sorted. FIFOE and LIFOE distribute load fractions in the order of decreasing computation speed (i.e., increasing value of computation parameter, E_k). SUMCE distributes and collects load fractions in the order of increasing value of the sum $C_k + E_k + \delta C_k$ (equivalent to sorting by the sum $C_k + E_k$).

To explore the effects of processor parameter values on the performance of the algorithms, several types of simulations were carried out as follows:

A1. *$\mathcal{C}$ homogeneous, $\mathcal{E}$ homogeneous*
Twenty-five different cases are considered, and the values for $\mathcal{C}$ and $\mathcal{E}$ in each case are obtained from the intervals $[C_{min}, C_{max}]$ and $[E_{min}, E_{max}]$ defined in Table 1. In each case, the intervals are partitioned into m equal-sized, contiguous, non-overlapping sub-intervals of size $(C_{max} - C_{min})/m$ and $(E_{max} - E_{min})/m$. If these sub-intervals are represented as $I_{c1}, \ldots, I_{cm}$ and $I_{e1}, \ldots, I_{em}$, then for each case, m^2 sub-cases can be defined by taking the Cartesian product of the inter-

Table 1 Parameters for simulation set A.

Case	$C_k \in$	$E_k \in$	Case	$C_k \in$	$E_k \in$
1	[1,10]	[1,10]	14	[10,100]	[1,100]
2	[1,10]	[10,100]	15	[10,100]	[10,1000]
3	[1,10]	[100,1000]	16	[10,1000]	[1,10]
4	[1,10]	[1,100]	17	[10,1000]	[10,100]
5	[1,10]	[10,1000]	18	[10,1000]	[100,1000]
6	[1,100]	[1,10]	19	[10,1000]	[1,100]
7	[1,100]	[10,100]	20	[10,1000]	[10,1000]
8	[1,100]	[100,1000]	21	[100,1000]	[1,10]
9	[1,100]	[1,100]	22	[100,1000]	[10,100]
10	[1,100]	[10,1000]	23	[100,1000]	[100,1000]
11	[10,100]	[1,10]	24	[100,1000]	[1,100]
12	[10,100]	[10,100]	25	[100,1000]	[10,1000]
13	[10,100]	[100,1000]			

vals, i.e. $\{(I_{c1}, I_{e1}), (I_{c1}, I_{e2}), \ldots, (I_{cm}, I_{em})\}$. For each sub-case, the intervals are uniformly sampled to generate the communication and computation parameters. Sampling the sub-intervals in this manner not only generates a homogeneous system, but also it is possible to compare a "fast" and (comparatively) "slow" homogeneous system.

A2. *C homogeneous, $\mathcal{E}$ heterogeneous*
Similar to Set A1, m sub-intervals $I_{c1}, \ldots, I_{cm}$ are used for the communication parameters, but the computation parameters are generated by sampling the interval $I_e = [E_{min}, E_{max}]$. This creates m sub-cases with intervals $\{(I_{c1}, I_e), \ldots, (I_{cm}, I_e)\}$, such that in each sub-case, the communication parameters are homogeneous, but the computation parameters are heterogeneous.

A3. *C heterogeneous, $\mathcal{E}$ homogeneous*
This is the complement of Set A2, and the m sub-cases have intervals $\{(I_c, I_{e1}), \ldots, (I_c, I_{em})\}$, where $I_c = [C_{min}, C_{max}]$, such that the computation parameters are homogeneous and the communication parameters are heterogeneous.

A4. *C heterogeneous, $\mathcal{E}$ heterogeneous*
Similar to the previous sets, the sub-intervals $I_{c1}, \ldots, I_{cm}$ and $I_{e1}, \ldots, I_{em}$ for each case are found. Each sub-interval is sampled once to generate a total of m values each for the communication and computation parameters. The values undergo a random permutation first before being assigned to the processors. Sampling the sub-intervals in this manner minimizes the possibility of two processors being allocated similar communication or computation parameters and generates a truly heterogeneous system. There are no sub-cases here.

For each set A1 to A4 mentioned above, simulations were carried out for $m = 4, 5$ and $\delta = 0.2, 0.5, 0.8$. For each variant algorithm, viz. OPT, SPORT, ITERLP, LIFOC, and FIFOC, at each value of m and δ, 100 simulation runs were carried out for each of the sub-cases in sets A1 to A3, and the 25 cases in Set A4. Let $v = 1, \ldots, 4$ represent the four variant algorithms SPORT, ITERLP, LIFOC, and FIFOC. The total processing time for each variant, T_v, was calculated in each run and the percentage deviation from the optimal processing time, ΔT_v, for each variant was calculated as:

$$\Delta T_v = \frac{T_v - T_{\text{OPT}}}{T_{\text{OPT}}} * 100\% \qquad v = 1, \ldots, 4$$

Further processing and analysis is explained individually for each set below. Plots are to log-scale to magnify the values close to zero. Not all plots are shown on account of space considerations.

A1. *C homogeneous, $\mathcal{E}$ homogeneous*
In this set, for each case in Table 1, there are 16 and 25 sub-cases for $m = 4$ and 5 respectively. The first sub-case represents a system with fast network links and high computation speed, while the last sub-case represents a system with slow network links and low computation speed. The intermediate sub-cases represent various other combinations of network and computation speed.
The cases in Table 1 are defined with intervals differing in both interval width (ratio) as well as the absolute values of the communication and computation parameters. To aggregate the performance obtained over all the intervals, the error values of the individual sub-cases should be averaged over the 25 cases in Table 1 to give the final mean percent error for each sub-case.
Let $i = 1, \ldots, 25$ represent the cases in Table 1, and $j = 1, \ldots, m^2$ represent the sub-cases in each case. Then for case i, sub-case j, 100 simulation runs generate 100 values ΔT^i_{vjk}, $k = 1, \ldots, 100$. These are averaged to get the mean error with respect to optimal, $\overline{\Delta T}^i_{vj}$, for each variant:

$$\overline{\Delta T}^i_{vj} = \frac{\sum_{k=1}^{100} \Delta T^i_{vjk}}{100} \qquad i = 1, \ldots, 25, \quad j = 1, \ldots, m^2, v = 1, \ldots, 4$$

Finally, for each variant, the values $\overline{\Delta T}^i_{vj}$ are averaged over the 25 cases in Table 1, to compute the final mean percent error, $\langle \overline{\Delta T}_{vj} \rangle$ for that variant in each sub-case:

$$\langle \overline{\Delta T}_{vj} \rangle = \frac{\sum_{i=1}^{25} \overline{\Delta T}^i_{vj}}{25} \quad j = 1, \ldots, m^2, v = 1, \ldots, 4$$

These values of $\langle \overline{\Delta T}_{vj} \rangle$ are plotted in Figs. 8 and 9 for (m, δ) pairs (4, 0.2) and (5,0.8) respectively.
Because the network links are homogeneous, FIFOC is expected to perform well. It is observed that SPORT performs almost exactly the same as FIFOC, with error between 0.1% and 0.01% $\approx$ 0. ITERLP performance is even better, while LIFOC has comparatively large error, and the error increases with increase in value of δ.

A2. *C homogeneous, $\mathcal{E}$ heterogeneous*
In this set, for each case in Table 1, there are 4 and 5 sub-cases for $m = 4$ and 5 respectively. The first sub-case represents a system with fast network links, while the last sub-case represents a system with slow network

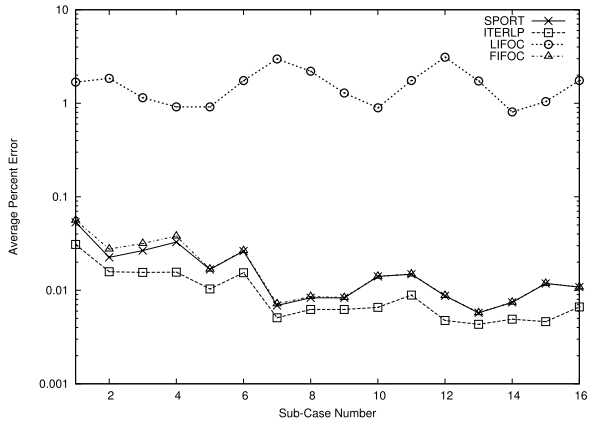


Fig. 8 $\langle \overline{\Delta T} \rangle$ in set A1 for $m = 4$, $\delta = 0.2$.

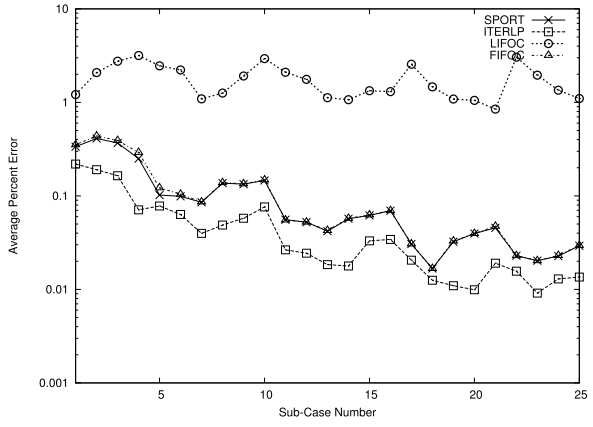


Fig. 9 $\langle \overline{\Delta T} \rangle$ in set A1 for $m = 5$, $\delta = 0.8$.

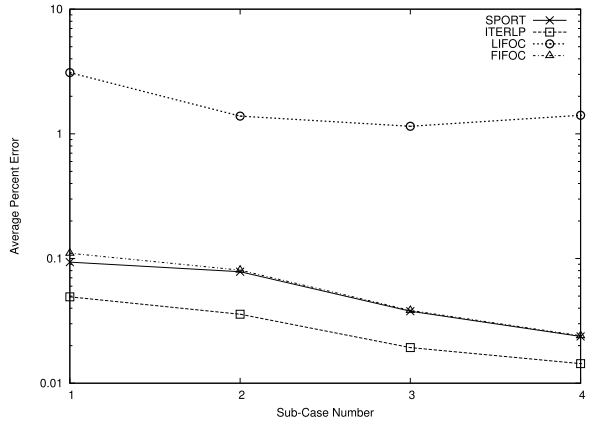


Fig. 10 $\langle \overline{\Delta T} \rangle$ in set A2 for $m = 4$, $\delta = 0.5$.

links. The computation speed is heterogeneous. The values of $\langle \overline{\Delta T}_{vj} \rangle$ for the sub-cases are calculated as explained in set A1 by averaging over the 25 cases in Table 1. The only difference is that the value of j ranges from $1, \ldots, m$ instead of from $1, \ldots, m^2$ as in set A1. Figs. 10 and 11 show the plots for (m, δ) pairs $(4, 0.5)$ and (5,0.2) respectively.

Again it is observed that as long as the network links

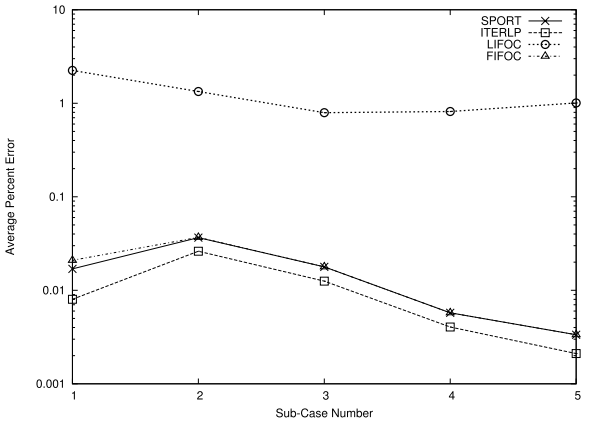


Fig. 11 $\langle \overline{\Delta T} \rangle$ in set A2 for $m = 5$, $\delta = 0.2$.

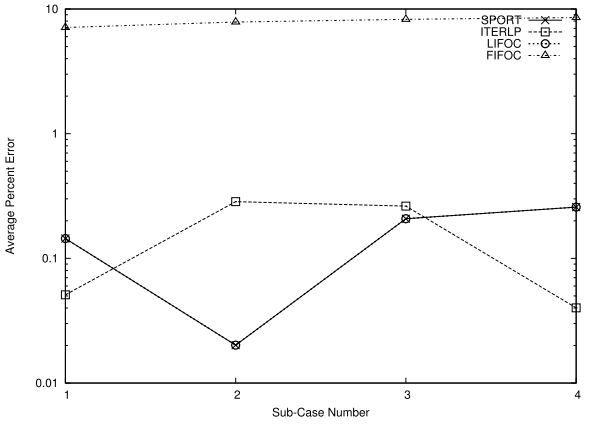


Fig. 12 $\langle \overline{\Delta T} \rangle$ in set A3 for $m = 4$, $\delta = 0.8$.

are homogeneous, SPORT, ITERLP, and FIFOC are insensitive to the heterogeneity in the computation speed of the processors with average error between 0.1% to $0.001\% \approx 0$. On the other hand, errors in LIFOC persist and increase with δ.

A3. *C heterogeneous, $\mathcal{E}$ homogeneous*

As in set A2, this set too has 4 and 5 sub-cases for $m = 4$ and 5 respectively. The first sub-case represents a system with fast computation speed, while the last sub-case represents a system with slow computation speed. The network links are heterogeneous. The values of $\langle \overline{\Delta T}_{vj} \rangle$ are calculated analogous to set A2. The plots for (m, δ) pairs $(4, 0.8)$ and $(5, 0.5)$ are shown in Figs. 12 and 13 respectively.

This simulation set clearly shows the adaptiveness of SPORT. FIFOC, which had almost zero error in the previous two sets, now has large error as compared to the optimal schedule. SPORT however, continues to have low error values around 0.1% along with LIFOC and ITERLP.

A4. *C heterogeneous, $\mathcal{E}$ heterogeneous*

Since there are no sub-cases in this set, for each case $i = 1, \ldots, 25$, in Table 1, 100 simulation runs generate 100 values ΔT^i_{vk}, $k = 1, \ldots, 100$. These 100 values are

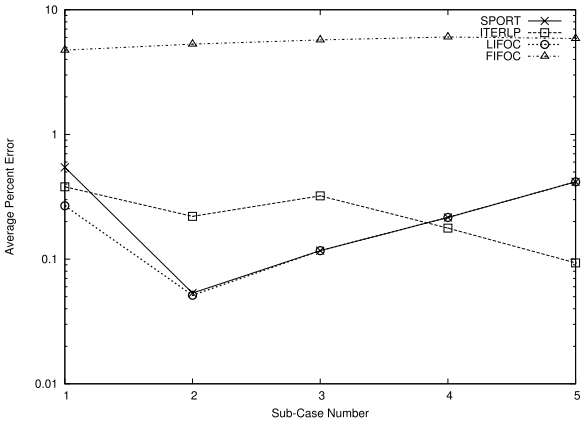


Fig. 13 $\langle \overline{\Delta T} \rangle$ in set A3 for $m = 5$, $\delta = 0.5$.

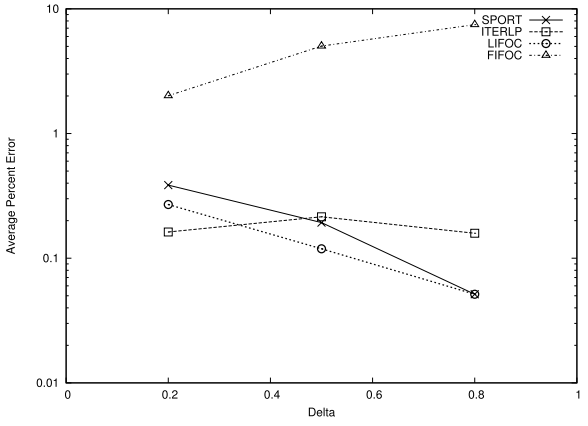


Fig. 14 $\langle \overline{\Delta T} \rangle$ in set A4 for $m = 4$.

averaged to compute the mean error from optimal $\overline{\Delta T}_v^i$:

$$\overline{\Delta T}_v^i = \frac{\sum_{k=1}^{100} \Delta T_{vk}^i}{100} \quad i = 1, \ldots, 25, v = 1, \ldots, 4$$

The final error values for each variant $\langle \overline{\Delta T}_v \rangle$ are calculated by averaging over the 25 cases in Table 1.

$$\langle \overline{\Delta T}_v \rangle = \frac{\sum_{i=1}^{25} \overline{\Delta T}_v^i}{25} \qquad v = 1, \ldots, 4$$

The values of $\langle \overline{\Delta T}_v \rangle$ for $\delta = 0.2, 0.5, 0.8$ at $m = 4$ and 5 are shown in Figs. 14 and 15 respectively.

SPORT, LIFOC, and ITERLP are seen adapt to the heterogeneity in the processor computation and network link speeds. The percent error of FIFOC increases with the increase in δ, but there is a reduction in the error of the other three variants.

Though not strictly applicable, some trends can be identified:

- In set A1, for the same value of δ, errors for "fast" homogeneous systems are higher than "slow" homogeneous systems.

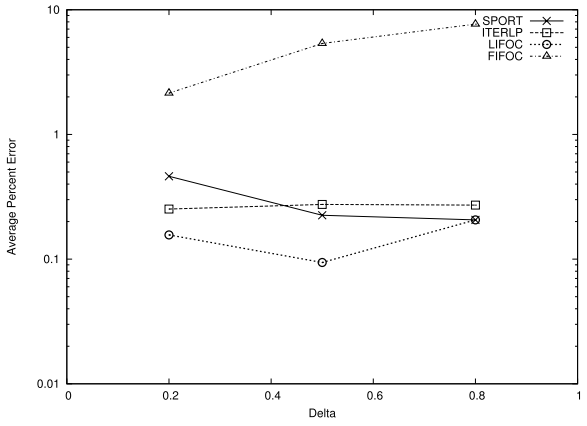


Fig. 15 $\langle \overline{\Delta T} \rangle$ in set A4 for $m = 5$.

- In sets A2 and A3, for the same value of δ, errors for the "fast" and heterogeneous systems are higher than the "slow" and heterogeneous systems.
- In sets A1, A2, and A3, where either one or both of the variables are homogeneous, the average error increases with increase in δ. However, in set A4, error reduces with δ, for the better performing algorithms.

The minimum and maximum mean error values of each algorithm are tabulated in Tables 2 and 3. Scientific notation is used to enable a quick comparison of the algorithms in terms of *orders of magnitude*. It can be observed that overall in sets A1 and A2, the minimum and maximum errors in LIFOC are 2 orders of magnitude higher than SPORT, ITERLP, and FIFOC. On the other hand in sets A3 and A4, FIFOC error is 2 to 3 orders of magnitude higher than the other three algorithms.

The extensive simulations carried out in Set A clearly show that:

- If network links are homogeneous, then LIFOC performance suffers for both homogeneous and heterogeneous computation speeds.
- If network links are heterogeneous, then FIFOC performance suffers for both homogeneous and heterogeneous computation speeds.
- SPORT performance is also affected to a certain degree by the heterogeneity in network links and computation speeds, but since SPORT does not use a single predefined sequence of allocation and collection, it is able to better adapt to the changing system conditions.
- ITERLP performance is somewhat better than SPORT, but is expensive to compute. SPORT generates similar schedules at a fraction of the cost.

5.2 Simulation Set B

To evaluate the performance of the algorithms with the increase in number of nodes, the processing times of FIFOC and LIFOC were compared with SPORT. This is because, OPT and ITERLP cannot be practically carried out beyond $m = 5$ and

Table 2 Minimum statistics for simulation set A.

Set	m	$\delta = 0.2$				$\delta = 0.5$				$\delta = 0.8$			
		SPORT	ITERLP	LIFOC	FIFOC	SPORT	ITERLP	LIFOC	FIFOC	SPORT	ITERLP	LIFOC	FIFOC
A1	4	5.73e-03	4.32e-03	8.08e-01	5.76e-03	2.20e-02	1.06e-02	1.07e+00	2.21e-02	3.58e-02	1.78e-02	1.16e+00	3.66e-02
	5	7.89e-04	6.90e-04	7.21e-01	7.89e-04	5.40e-03	4.21e-03	9.63e-01	5.30e-03	1.67e-02	9.13e-03	8.47e-01	1.67e-02
A2	4	1.01e-02	5.78e-03	8.41e-01	1.01e-02	2.37e-02	1.43e-02	1.15e+00	2.40e-02	3.59e-02	2.06e-02	1.22e+00	3.71e-02
	5	3.34e-03	2.10e-03	7.93e-01	3.34e-03	1.06e-02	8.92e-03	1.10e+00	1.07e-02	2.01e-02	1.69e-02	1.06e+00	2.02e-02
A3	4	2.03e-01	1.80e-03	1.05e-01	1.61e+00	1.12e-01	5.13e-03	9.59e-02	4.43e+00	2.01e-02	4.01e-02	2.01e-02	7.11e+00
	5	3.96e-01	1.90e-01	8.90e-02	1.75e+00	5.34e-02	9.32e-02	5.13e-02	4.74e+00	5.14e-02	4.98e-03	5.14e-02	7.42e+00
A4	4	4.95e-06	1.97e-16	4.92e-06	1.05e+00	3.09e-02	2.77e-15	3.09e-02	3.23e+00	6.15e-02	4.01e-03	6.15e-02	5.58e+00
	5	1.08e-02	5.81e-04	2.75e-06	1.15e+00	5.84e-02	2.18e-03	5.84e-02	3.74e+00	9.43e-11	0.00e+00	7.05e-11	6.38e-01

Table 3 Maximum statistics for simulation set A.

Set	m	$\delta = 0.2$				$\delta = 0.5$				$\delta = 0.8$			
		SPORT	ITERLP	LIFOC	FIFOC	SPORT	ITERLP	LIFOC	FIFOC	SPORT	ITERLP	LIFOC	FIFOC
A1	4	5.34e-02	3.09e-02	3.11e+00	5.61e-02	1.84e-01	7.57e-02	4.20e+00	2.02e-01	2.57e-01	1.13e-01	3.39e+00	3.08e-01
	5	8.24e-02	4.87e-02	3.00e+00	8.79e-02	2.26e-01	1.19e-01	3.91e+00	2.30e-01	4.10e-01	2.19e-01	3.17e+00	4.37e-01
A2	4	3.03e-02	1.69e-02	1.83e+00	3.06e-02	9.35e-02	4.93e-02	3.10e+00	1.10e-01	2.44e-01	1.10e-01	2.91e+00	2.79e-01
	5	3.66e-02	2.61e-02	2.24e+00	3.68e-02	1.15e-01	8.34e-02	2.75e+00	1.26e-01	2.72e-01	1.27e-01	2.84e+00	2.89e-01
A3	4	4.01e-01	3.42e-01	4.66e-01	2.02e+00	4.03e-01	2.22e-01	4.03e-01	5.44e+00	2.57e-01	2.85e-01	2.57e-01	8.53e+00
	5	5.31e-01	3.86e-01	4.84e-01	2.30e+00	5.45e-01	3.80e-01	4.16e-01	6.05e+00	2.55e-01	4.37e-01	2.55e-01	9.22e+00
A4	4	1.32e+00	6.50e-01	8.84e-01	4.47e+00	8.02e-01	7.11e-01	4.00e-01	1.12e+01	1.56e-01	6.26e-01	1.56e-01	1.64e+01
	5	1.56e+00	7.66e-01	4.34e-01	4.85e+00	9.35e-01	8.97e-01	4.24e-01	1.15e+01	1.36e+00	2.04e+00	1.36e+00	1.63e+01

$m = 10$ respectively. Using the procedure used in simulation Set A4, 100 simulation runs were carried out for SPORT, LIFOC, and FIFOC, at $m = 10, 50, 100, \ldots, 300, 350$, and $\delta = 0.2, 0.5, 0.8$ for each of the 25 cases listed in Table 1. ΔT_v, for each variant v (LIFOC := 1 and FIFOC := 2) was found as:

$$\Delta T_v = \frac{T_v - T_{\text{SPORT}}}{T_{\text{SPORT}}} * 100\% \qquad v = 1, 2$$

Mean error, $\overline{\Delta T}_v^i$, for each case $i = 1, \ldots, 25$ in Table 1 was calculated by averaging ΔT_{vk}^i, $k = 1, \ldots, 100$, over the 100 simulation runs and plotted.

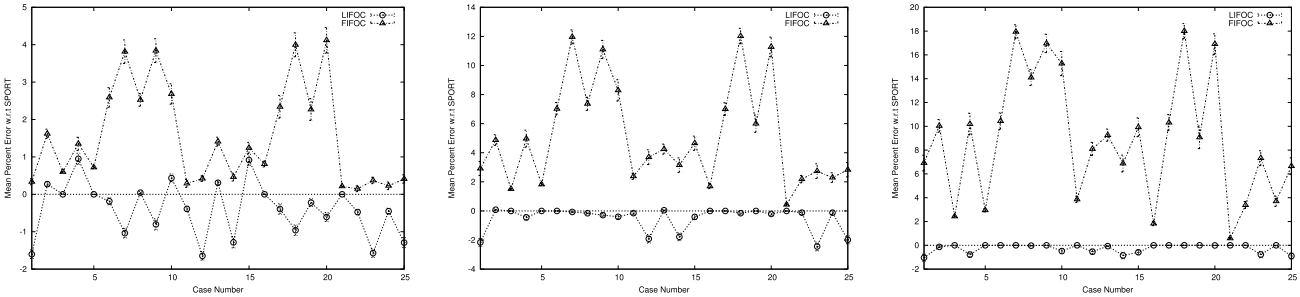
Figure 16 shows the plots for $\overline{\Delta T}_v^i$ at $m = 10$, $\delta = 0.2, 0.5, 0.8$. First of all, FIFOC is seen to always have a positive error with respect to SPORT. This is to be expected since the system is heterogeneous. The value of error increases with increase in the value of δ.

Secondly, LIFOC has a negative error with respect to SPORT for several cases at $\delta = 0.2$, i.e. the processing time of LIFOC is smaller than SPORT. This is also to be expected since LIFOC uses all available processors and every processor added reduces the processing time by some amount. This ends up distributing very tiny load fractions (smaller than 1×10^{-6}) to a large number of tail-end processors. As the value of δ increases, the error between LIFOC and SPORT becomes insignificant.

This pattern of results is repeated even for higher values of m as can be seen in Figs. 17 and 18 for $m = 100$ and 300 respectively. It can be observed that as the number of processors increases, FIFOC performance in case numbers 11–15 and 21–25 becomes almost equal to that of SPORT. These ranges correspond to the intervals $I_c = [10, 100]$ and $I_c = [100, 1000]$ respectively, i.e., a ratio of $C_{min} : C_{max} = 1 : 10$. Because of the methodology used to perform the simulations, with a large number of processors, the values of C_k tend to become similar to each other. Consequently, $(C_2 - C_1)$ in (A· 18) becomes small, and Schedule f always tends to be optimal for each pair of processors being compared. If Schedule f is optimal for all processors in SPORT, the resulting σ_a and σ_c are the same as FIFOC. However, surprisingly, cases 1–5, with $I_c = [1, 10]$ do not show this trend. This leads us to hypothesize, that the performance of the algorithms not only depends on the range (ratio) of parameters but also on the absolute values of the parameters. This belief is reinforced by the fact that in case numbers 21–25, LIFOC also has comparable performance to SPORT, especially at higher values of δ and m.

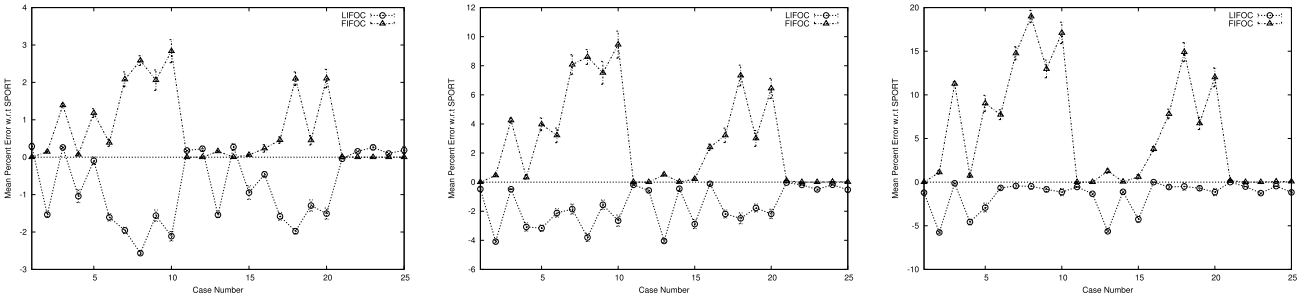
Consider Table 4 that gives the minimum error of LIFOC with respect to SPORT, the case number when it occurs, along with the mean error of LIFOC averaged over all 25 cases (i.e., $\langle \overline{\Delta T}_v \rangle$ in set A4) for different values of δ and m. The minimum error for LIFOC is -5.76% for $m = 100$, $\delta = 0.8$, case number 2, but the minimum average error is -2.12% for $m = 300$, $\delta = 0.5$. It can be observed that the average error values at $\delta = 0.5$ are all smaller than those at $\delta = 0.2$, while the average error values at $\delta = 0.8$ are again greater than those at $\delta = 0.5$ (except for $m = 250$). We hypothesize that initially as δ increases, the error increases, but as $\delta \to 1$, i.e., size of result data approaches the size of allocated load, performance of SPORT and LIFOC becomes similar. This is supported by the results of set A4, where the $\langle \overline{\Delta T}_v \rangle$ of SPORT and LIFOC is almost equal for $\delta = 0.8$ (see Figs. 14 and 15).

There is a significant downside to LIFOC because of its property to use all available processors — the time required to compute the optimal solution (CPU time) is more than an order of magnitude greater than that of SPORT as seen in Fig. 19. These values were obtained separately from the



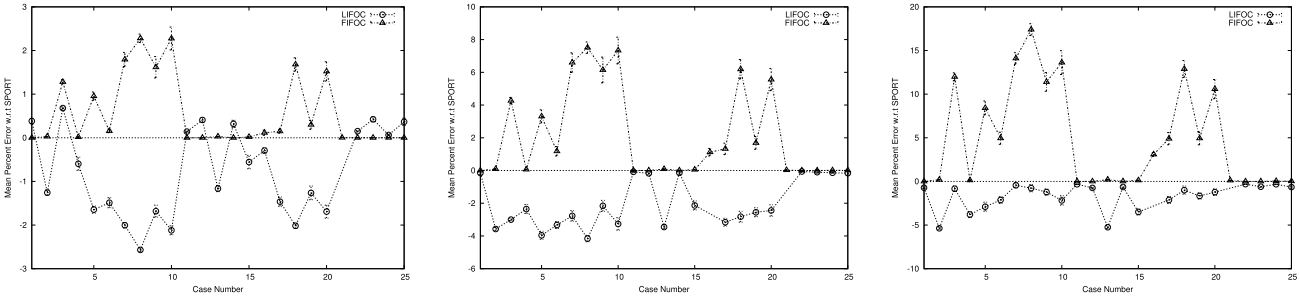
(a) $\overline{\Delta T}$ at $\delta = 0.2$ (b) $\overline{\Delta T}$ at $\delta = 0.5$ (c) $\overline{\Delta T}$ at $\delta = 0.8$

Fig. 16 Simulation set B, m = 10.



(a) $\overline{\Delta T}$ at $\delta = 0.2$ (b) $\overline{\Delta T}$ at $\delta = 0.5$ (c) $\overline{\Delta T}$ at $\delta = 0.8$

Fig. 17 Simulation set B, m = 100.



(a) $\overline{\Delta T}$ at $\delta = 0.2$ (b) $\overline{\Delta T}$ at $\delta = 0.5$ (c) $\overline{\Delta T}$ at $\delta = 0.8$

Fig. 18 Simulation set B, m = 300.

Table 4 Statistics for LIFOC in simulation set B.

m	$\delta = 0.2$			$\delta = 0.5$			$\delta = 0.8$		
	case	min	avg	case	min	avg	case	min	avg
10	12	−1.64	−0.39	23	−2.44	−0.50	1	−1.04	−0.24
50	8	−2.41	−0.88	2	−4.39	−1.47	4	−3.72	−1.33
100	8	−2.56	−0.79	2	−4.08	−1.66	2	−5.76	−1.49
150	8	−2.56	−0.78	8	−4.16	−2.01	2	−5.37	−1.68
200	8	−2.57	−0.82	5	−4.25	−2.06	13	−4.55	−1.77
250	8	−2.55	−0.77	17	−4.28	−1.36	17	−4.01	−1.85
300	8	−2.54	−0.88	3	−4.57	−2.12	5	−4.47	−1.70
350	8	−2.52	−0.83	3	−4.63	−2.04	5	−4.53	−1.67

simulations above by averaging the CPU time over 100 runs for $I_c = [10, 100]$, $I_e = [50, 500]$, and $\delta = 0.5$. The results show that though both SPORT and LIFOC are $O(m)$ algorithms, clearly SPORT is the better performing algorithm, with the best cost/performance ratio for large values of m. The CPU time values for FIFOC are more than three orders of magnitude larger than SPORT — too large to even warrant consideration.

The other disadvantage of LIFOC is that the chain of multiplications involved in the calculation of load fractions quickly leads to underflow because the numbers involved are tiny fractions and multiplying them results in smaller and smaller numbers until the floating point system cannot handle them anymore. Because of this, for $m > 150$, it is difficult to get valid results for LIFOC in a large number of cases. For example, for m = 250, 300, 350, LIFOC returned underflow errors in 24, 32 and 32 runs respectively out of the 100 simulation runs carried out. In MATLAB™, this causes a NaN (Not A Number) to be returned, and the load fractions cannot be calculated. Of course this is not a limitation of the algorithm itself, nevertheless it is an important practical consideration during implementation.

m	SPORT	LIFOC	FIFOC
50	0.00025	0.00427	0.119
100	0.00056	0.00687	0.469
150	0.00063	0.01038	1.319
200	0.00071	0.01409	2.874
250	0.00077	0.01743	5.399
300	0.00084	0.02112	9.110
350	0.00092	0.02509	14.046
400	0.00099	0.02951	20.419
450	0.00107	0.03458	28.482
500	0.00114	0.04018	37.497

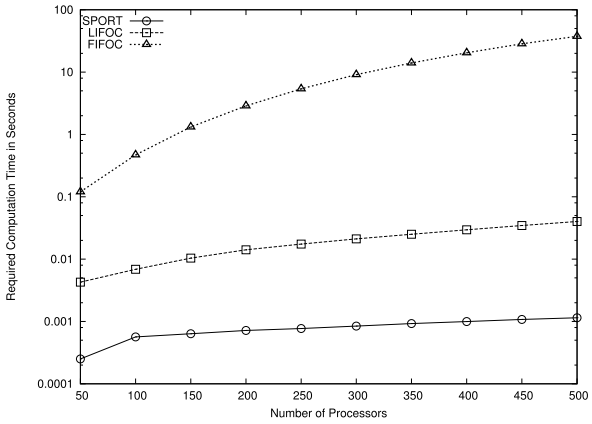


Fig. 19 Comparison of CPU time for SPORT, LIFOC, and FIFOC.

6. Conclusion

In this paper, a polynomial time algorithm, SPORT, for the scheduling of divisible loads on heterogeneous processing platforms and considering the result collection phase is presented. A large number of simulations were performed and it is found that SPORT consistently delivers good performance irrespective of the degree of heterogeneity of the system, the number of nodes, or the size of result data. SPORT primarily uses LIFOC and FIFOC as its base, but instead of having a single predefined sequence, it iteratively builds a locally optimal solution leading to a low error value.

This paper includes, for the first time ever, (a) the derivation of the condition to identify the presence of idle time in a FIFO schedule for two processors, (b) the identification of the limiting condition for the optimality of FIFO and LIFO schedules for two processors, and (c) the introduction of the concept of *equivalent processor* in DLSRCHETS.

As future work, the conditions (constraints on values of E_k and C_k), that minimize the error need to be found. An interesting area would be the investigation of the effect of affine cost models, processor deadlines and release times. Another important area would be to extend the results to multi-installment delivery and multi-level processor trees.

Acknowledgments

The authors would like to sincerely thank the reviewers for their insightful comments on the previous drafts of this paper.

References

[1] V. Bharadwaj, D. Ghose, V. Mani, and T.G. Robertazzi, Scheduling Divisible Loads in Parallel and Distributed Systems, IEEE Computer Society Press, Los Alamitos, CA, 1996.

[2] V. Bharadwaj, D. Ghose, and T.G. Robertazzi, "Divisible load theory: A new paradigm for load scheduling in distributed systems," Cluster Computing, vol.6, no.1, pp.7–17, Jan. 2003.

[3] T. Robertazzi, http://www.ece.sunysb.edu/~tom/dlt.html, April 2005.

[4] N. Comino and V.L. Narasimhan, "A novel data distribution technique for host-client type parallel applications," IEEE Trans. Parallel Distrib. Syst., vol.13, no.2, pp.97–110, Feb. 2002.

[5] V. Bharadwaj, X. Li, and C.C. Ko, "Efficient partitioning and scheduling of computer vision and image processing data on bus networks using divisible load analysis," Image Vis. Comput., vol.18, no.1, pp.919–938, Jan. 2000.

[6] V. Bharadwaj, D. Ghose, and V. Mani, "Optimal sequencing and arrangement in distributed single-level tree networks with communication delays," IEEE Trans. Parallel Distrib. Syst., vol.5, no.9, pp.968–976, Sept. 1994.

[7] V. Bharadwaj, D. Ghose, and V. Mani, "Multi-installment load distribution in tree networks with delays," IEEE Trans. Aerosp. Electron. Syst., vol.31, no.2, pp.555–567, April 1995.

[8] V. Bharadwaj, X. Li, and C.C. Ko, "On the influence of start-up costs in scheduling divisible loads on bus networks," IEEE Trans. Parallel Distrib. Syst., vol.11, no.12, pp.1288–1305, Dec. 2000.

[9] X. Li, V. Bharadwaj, and C.C. Ko, "Divisible load scheduling on single-level tree networks with buffer constraints," IEEE Trans. Aerosp. Electron. Syst., vol.36, no.4, pp.1298–1308, Oct. 2000.

[10] S. Bataineh, T.Y. Hsiung, and T.G. Robertazzi, "Closed form solutions for bus and tree networks of processors load sharing a divisible job," IEEE Trans. Comput., vol.43, no.10, pp.1184–1196, Oct. 1994.

[11] S. Bataineh and B. Al-Asir, "An efficient scheduling algorithm for divisible and indivisible tasks in loosely coupled multiprocessor systems," Software Engineering J., vol.9, no.1, pp.13–18, Jan. 1994.

[12] S. Bataineh and T.G. Robertazzi, "Performance limits for processors with divisible jobs," IEEE Trans. Aerosp. Electron. Syst., vol.33, no.4, pp.1189–1198, Oct. 1997.

[13] J. Sohn and T.G. Robertazzi, "Optimal divisible job load sharing for bus networks," IEEE Trans. Aerosp. Electron. Syst., vol.32, no.1, pp.34–40, Jan. 1996.

[14] J. Sohn, T.G. Robertazzi, and S. Luryi, "Optimizing computing costs using divisible load analysis," IEEE Trans. Parallel Distrib. Syst., vol.9, no.3, pp.225–234, March 1998.

[15] T.G. Robertazzi, "Processor equivalence for daisy chain load sharing processors," IEEE Trans. Aerosp. Electron. Syst., vol.29, no.4, pp.1216–1221, Oct. 1993.

[16] Y.C. Cheng and T.G. Robertazzi, "Distributed computation for a tree network with communication delays," IEEE Trans. Aerosp. Electron. Syst., vol.26, no.3, pp.511–516, May 1990.

[17] G.D. Barlas, "Collection-aware optimum sequencing of operations and closed-form solutions for the distribution of a divisible load on arbitrary processor trees," IEEE Trans. Parallel Distrib. Syst., vol.9, no.5, pp.429–441, May 1998.

[18] V. Mani and D. Ghose, "Distributed computation in linear networks: Closed-form solutions," IEEE Trans. Aerosp. Electron. Syst., vol.30,

no.2, pp.471–483, April 1994.
[19] D. Ghose and V. Mani, "Distributed computation with communication delays: Asymptotic performance analysis," J. Parallel Distrib. Comput., vol.23, no.3, pp.293–305, Dec. 1994.
[20] D. Ghose and H.J. Kim, "Load partitioning and trade-off study for large matrix-vector computations in multicast bus networks with communication delays," J. Parallel Distrib. Comput., vol.55, no.1, pp.32–59, Nov. 1998.
[21] H.J. Kim, G. Jee, and J.G. Lee, "Optimal load distribution for tree network processors," IEEE Trans. Aerosp. Electron. Syst., vol.32, no.2, pp.607–612, April 1996.
[22] C.H. Lee and K.G. Shin, "Optimal task assignment in homogeneous networks," IEEE Trans. Parallel Distrib. Syst., vol.8, no.2, pp.119–129, Feb. 1997.
[23] O. Beaumont, L. Marchal, and Y. Robert, "Scheduling divisible loads with return messages on heterogeneous master-worker platforms," LIP, ENS Lyon, France, Research Report 2005-21, May 2005.
[24] O. Beaumont, L. Marchal, V. Rehn, and Y. Robert, "FIFO scheduling of divisible loads with return messages under the one-port model," LIP, ENS Lyon, France, Research Report 2005-52, Oct. 2005.
[25] O. Beaumont, L. Marchal, V. Rehn, and Y. Robert, "FIFO scheduling of divisible loads with return messages under the one port model," Proc. Heterogeneous Computing Workshop HCW'06, April 2006.
[26] A. Rosenberg, "Sharing partitionable workload in heterogeneous NOWs: Greedier is not better," IEEE International Conference on Cluster Computing, pp.124–131, Newport Beach, CA, Oct. 2001.
[27] R.J. Vanderbei, Linear Programming: Foundations and Extensions, 2nd ed., International Series in Operations Research & Management, vol.37, Kluwer Academic Publishers, 2001.
[28] A. Galstyan, K. Czajkowski, and K. Lerman, "Resource allocation in the grid using reinforcement learning," Intl. Jt. Conf. on Autonomous Agents and Multiagent Systems, pp.1314–1315, 2004.

Appendix: Derivations

The equations for load fractions, processing times, and the conditions for optimality of Schedules f, l, and g are derived in brief in A.1, A.2, and A.3. Several intermediate steps in the derivations are not shown on account of space considerations.

A.1 Schedule f

From Fig. 4(a), $\alpha_1^f(E_1+\delta C_1) = \alpha_2^f(C_2+E_2)$. Using $\alpha_1^f+\alpha_2^f = 1$ gives

$$\alpha_1^f = \frac{C_2 r_2^f}{C_1 r_1^f + C_2 r_2^f} \tag{A·1}$$

$$\alpha_2^f = \frac{C_1 r_1^f}{C_1 r_1^f + C_2 r_2^f} \tag{A·2}$$

where

$$r_1^f = \delta + \rho_1$$
$$r_2^f = 1 + \rho_2$$

It is interesting to see α_1^f and α_2^f as weighted ratios of C_2 and C_1. The weights are functions of the network as well as the application under consideration. The processing time of Schedule f is

$$T^f = \alpha_1^f C_1(1+\delta+\rho_1) + \delta\alpha_2^f C_2$$

Using (A·1) and (A·2),

$$= \frac{C_1 C_2}{C_1 r_1^f + C_2 r_2^f}\Big((1+r_1^f)(\delta+r_2^f) - \delta\Big) \tag{A·3}$$

It is advantageous to distribute load to the two processors in Schedule f instead of processing it entirely on p_1, if $T^f \le T^1$. Now, from (A·3),

$$T^f \le T^1 \Leftrightarrow \frac{C_1 C_2}{C_1 r_1^f + C_2 r_2^f}(\delta r_1^f + r_2^f + r_1^f r_2^f) \le C_1(1+r_1^f)$$
$$\Leftrightarrow \delta C_2 \le C_1(1+\delta+\rho_1) \tag{A·4}$$

Similarly, it is advantageous to distribute load to the two processors in Schedule f instead of processing it entirely on p_2, if $T^f \le T^2$. Again using (A·3),

$$T^f \le T^2 \Leftrightarrow \frac{C_1 C_2}{C_1 r_1^f + C_2 r_2^f}(\delta r_1^f + r_2^f + r_1^f r_2^f) \le C_2(\delta + r_2^f)$$
$$\Leftrightarrow \delta C_1 \le C_2(1+\delta+\rho_2) \tag{A·5}$$

Equation (A·5) is always true for $C_1 \le C_2$.

To derive the equation for equivalent processor of Schedule f, algebraic manipulation of (A·3) gives

$$T_{1:2}^f = \frac{C_1 C_2}{C_1 r_1^f + C_2 r_2^f}\Big(r_1^f + r_2^f + \delta r_1^f + \delta r_2^f + r_1^f r_2^f - r_1^f - \delta r_2^f + \delta - \delta\Big)$$
$$= \frac{C_1 C_2(r_1^f + r_2^f)}{C_1 r_1^f + C_2 r_2^f} + \frac{C_1 C_2(\rho_1\rho_2 - \delta)}{C_1 r_1^f + C_2 r_2^f} + \frac{\delta C_1 C_2(r_1^f + r_2^f)}{C_1 r_1^f + C_2 r_2^f} \tag{A·6}$$

A.2 Schedule l

From Fig. 4(b), $\alpha_1^l E_1 = \alpha_2^l(C_2+E_2+\delta C_2)$. Using $\alpha_1^l+\alpha_2^l = 1$ gives

$$\alpha_1^l = \frac{C_2 r_2^l}{C_1 r_1^l + C_2 r_2^l} \tag{A·7}$$

$$\alpha_2^l = \frac{C_1 r_1^l}{C_1 r_1^l + C_2 r_2^l} \tag{A·8}$$

where

$$r_1^l = \rho_1 \quad (A\cdot 9)$$
$$r_2^l = 1 + \delta + \rho_2 \quad (A\cdot 10)$$

We note that

$$r_1^l = r_1^f - \delta \quad (A\cdot 11)$$
$$r_2^l = r_2^f + \delta \quad (A\cdot 12)$$
$$r_1^l + r_2^l = r_1^f + r_2^f = 1 + \delta + \rho_1 + \rho_2 \quad (A\cdot 13)$$

The processing time of Schedule l is

$$T^l = \alpha_1^l C_1(1 + \delta + \rho_1)$$

Using (A· 7),

$$= \frac{C_1 C_2 r_2^l (1 + \delta + r_1^l)}{C_1 r_1^l + C_2 r_2^l} \quad (A\cdot 14)$$

Using (A· 11) and (A· 12)

$$= \frac{C_1 C_2 (1 + r_1^f)(\delta + r_2^f)}{C_1 r_1^f + C_2 r_2^f + \delta(C_2 - C_1)} \quad (A\cdot 15)$$

Schedule l is advantageous instead of distributing load entirely to p_1, if $T^l \le T^1$. From (A· 14),

$$T^l \le T^1 \Leftrightarrow \frac{C_1 C_2 r_2^l (1+\delta+r_1^l)}{C_1 r_1^l + C_2 r_2^l} \le C_1(1+\delta+r_1^l) \Leftrightarrow 0 \le C_1\rho_1 \quad (A\cdot 16)$$

Equation (A· 16) is always true. Similarly, Schedule l is advantageous as compared to processing entire load on p_2, if $T^l \le T^2$. Again using (A· 14),

$$T^l \le T^2 \Leftrightarrow \frac{C_1 C_2 r_2^l (1 + \delta + r_1^l)}{C_1 r_1^l + C_2 r_2^l} \le C_2 r_2^l \Leftrightarrow C_1(1 + \delta) \le C_2(1 + \delta + \rho_2) \quad (A\cdot 17)$$

Equation (A· 17) is always true for $C_1 \le C_2$.

To find the limiting condition for the optimality of Schedules f and l, the equations for T^f and T^l are compared. From (A· 3) and (A· 15),

$$T^f \le T^l \Leftrightarrow \frac{(1 + r_1^f)(\delta + r_2^f) - \delta}{C_1 r_1^f + C_2 r_2^f} \le \frac{(1 + r_1^f)(\delta + r_2^f)}{C_1 r_1^f + C_2 r_2^f + \delta(C_2 - C_1)}$$
$$\Leftrightarrow \frac{C_1 C_2}{C_1 r_1^f + C_2 r_2^f}\Big((1 + r_1^f)(\delta + r_2^f) - \delta\Big) \le \frac{C_1 C_2}{(C_2 - C_1)}$$
$$\Leftrightarrow T^f \le \frac{C_1 C_2}{(C_2 - C_1)} \quad (A\cdot 18)$$

Conversely, it can be easily proved that

$$T^l \ge \frac{C_1 C_2}{(C_2 - C_1)} \Leftrightarrow T^l \ge T^f$$

To derive the equation for equivalent processor for Schedule l, algebraic manipulation of (A· 15) gives

$$T^l_{1:2} = \frac{C_1 C_2}{C_1 r_1^f + C_2 r_2^f + \delta(C_2 - C_1)}\Big(r_1^f + r_2^f + \delta r_1^f + \delta r_2^f + r_1^f r_2^f - r_1^f - \delta r_2^f + \delta\Big)$$

Using (A· 11), (A· 12), and (A· 13),

$$= \frac{C_1 C_2 (r_1^l + r_2^l)}{C_1 r_1^l + C_2 r_2^l} + \frac{C_1 C_2 \rho_1 \rho_2}{C_1 r_1^l + C_2 r_2^l} + \frac{\delta C_1 C_2 (r_1^l + r_2^l)}{C_1 r_1^l + C_2 r_2^l} \quad (A\cdot 19)$$

A.3 Schedule g

From Fig. 4(c), $\alpha_1^g E_1 = \alpha_2^g C_2$. Using $\alpha_1^g + \alpha_2^g = 1$ gives

$$\alpha_1^g = \frac{C_2}{E_1 + C_2} = \frac{C_2}{C_1\rho_1 + C_2} \quad (A\cdot 20)$$
$$\alpha_2^g = \frac{E_1}{E_1 + C_2} = \frac{C_1\rho_1}{C_1\rho_1 + C_2} \quad (A\cdot 21)$$

Idle time occurs in processor p_2 (i.e., $x_2 \ge 0$), only when $\alpha_2^g E_2 \le \delta\alpha_1^g C_1$. From (A· 20) and (A· 21),

$$\alpha_2^g E_2 \le \delta\alpha_1^g C_1 \Leftrightarrow \rho_1\rho_2 \le \delta \quad (A\cdot 22)$$

The processing time of Schedule g is

$$T^g = \alpha_1^g C_1(1 + \delta + \rho_1) + \delta\alpha_2^g C_2$$

Using (A· 20) and (A· 21),

$$= \frac{C_1 C_2}{C_1\rho_1 + C_2}(1 + \delta)(1 + \rho_1) \quad (A\cdot 23)$$

It is advantageous to distribute load to the two processors in Schedule g instead of processing it entirely on p_1, if $T^g \le T^1$. From (A· 23),

$$T^g \le T^1 \Leftrightarrow \frac{C_1 C_2}{C_1\rho_1 + C_2}(1 + \delta)(1 + \rho_1) \le C_1(1 + \delta + \rho_1) \Leftrightarrow \delta C_2 \le C_1(1 + \delta + \rho_1) \quad (A\cdot 24)$$

Similarly, it is advantageous to distribute load to the two processors in Schedule g instead of processing it entirely on p_2, if $T^g \le T^2$. Again using (A· 23),

$$T^g \le T^2 \Leftrightarrow \frac{C_1 C_2}{C_1\rho_1 + C_2}(1 + \delta)(1 + \rho_1) \le C_2(1 + \delta + \rho_2) \Leftrightarrow C_1(1 + \delta - \rho_1\rho_2) \le C_2(1 + \delta + \rho_2) \quad (A\cdot 25)$$

Equation (A· 25) is always true for $C_1 \leq C_2$.

To find the limiting condition for optimality of Schedules g and l, the equations for T^g and T^l are compared. From (A· 23) and (A· 14),

$$T^g \leq T^l \Leftrightarrow \frac{(1+\delta)(1+\rho_1)}{C_1\rho_1 + C_2} \leq \frac{(1+\delta+r_1^l)r_2^l}{C_1 r_1^l + C_2 r_2^l}$$

Using (A· 9),

$$\Leftrightarrow C_2 r_2^l \delta\rho_1 \leq C_1\rho_1(\delta r_2^l + \rho_2 + \rho_1\rho_2)$$

Using (A· 10),

$$\Leftrightarrow C_2 \leq C_1\Big(1 + \frac{(1+\rho_1)\rho_2}{\delta(1+\delta+\rho_2)}\Big) \qquad \text{(A· 26)}$$

Abhay Ghatpande received his B.E. degree from University of Pune, India in 1997, and his M.S. degree from Waseda University, Tokyo in 2004. He is presently a Research Associate and Ph.D. candidate there. In 1997 he started his career in Larsen & Toubro Ltd., as software engineer. In 2000, he helped set up MoTech Software's Japan branch office. In 2002 he turned to research. His research interests include parallel and distributed computing, multi-agent systems, and use of inference and learning algorithms in high performance computing. He is a member of the IEEE.

Hidenori Nakazato received his B. Engineering degree in Electronics and Telecommunications from Waseda University in 1982, and his M.S. and Ph.D. degrees in Computer Science from University of Illinois in 1989 and 1993, respectively. He was with Oki Electric from 1982 to 2000 where he developed equipment for public telephony switches, distributed environment for telecommunications systems, and communications quality control mechanisms. He has been a Professor at Graduate School of Global Information and Telecommunications Studies, Waseda University since 2000. His research interests include performance issues in distributed systems and networks, cooperation mechanisms of distributed programs, distributed real-time systems, and network QoS control.

Olivier Beaumont received his Ph.D. from the University of Rennes in 1999. From 1999 to 2001, he was assistant professor at Ecole Normale Supérieure de Lyon. He was appointed at ENSEIRB in Bordeaux. In 2004, he defended his "habilitation à diriger les recherches." He is the author of more than 15 papers published in international journals and 50 papers published in international conferences. His research interests are design of parallel, distributed and randomized algorithms, overlay networks on large scale heterogeneous platforms and combinatorial optimization.

Hiroshi Watanabe received the B.E., M.E. and Ph.D. degrees from Hokkaido University, Japan, in 1980, 1982 and 1985, respectively. He joined Nippon Telegraph and Telephone Corporation (NTT) in 1985, and engaged in R&D for image and video coding systems at NTT Human Interface Labs. (Later NTT Cyber Space Labs.) until August 2000. He has also engaged in developing JPEG, MPEG standards under JTC 1/SC 29. From Sept. 2000, he is a Professor at Graduate School of Global Information and Telecommunication Studies. He has been ISO/IEC JTC 1/SC 29 Chairman since November 1999. He is an associate editor of IEEE Transactions of Circuits and Systems for Video Technology. He is a member of a steering committee of PCSJ. He is a member of IEEE, IPSJ, ITE, IIEEJ.