信号理論

- No.8 ウェーブレット -

渡辺　裕

Signal Theory

- No.8 Wavelet -

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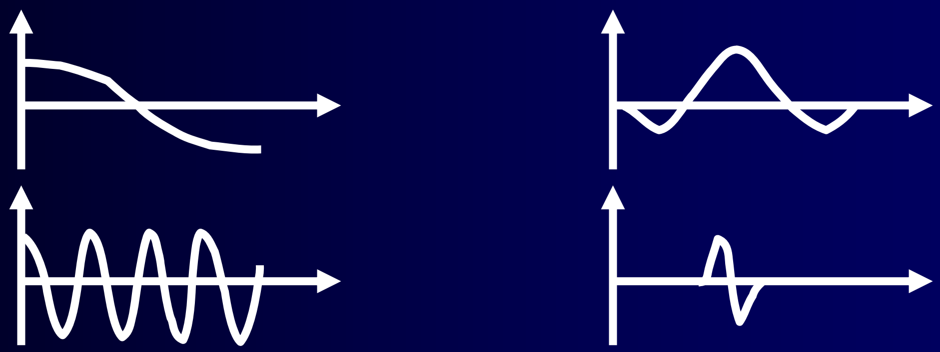
ウェーブレット

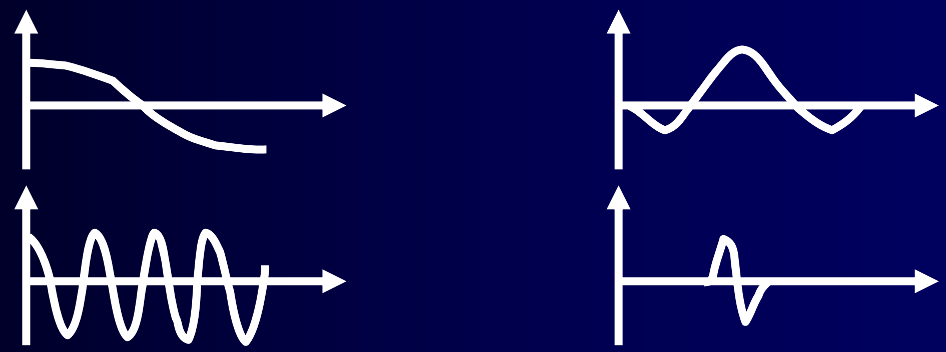
- ウェーブレットとは，基本フィルタの周波数解像度と時間解像度を変化されたフィルタ群を用いて，信号を表現する手法
- DFT(離散フーリエ変換)は長時間の信号を観察し，高い周波数解像度により信号を分析しようとする手法
- サブバンド分析は，一定の時間解像度および一定の周波数解像度を持つフィルタ群を用いて，信号を表現する手法



フーリエ変換　　ウェーブレット変換

Wavelet

- Wavelet is a technique to describe signal using a set of filters having varied frequency and time axis resolutions based on a basic filter
- DFT(Discrete Fourier Transform) is a technique to analyze signal using maximum frequency resolution in a long term
- Subband analysis is a technique to describe signal using filters having certain frequency and time axis resolution



Fourier Transform Wavelet Transform

ウェーブレット (2)

- 連続系のウェーブレット信号
  - dilation: スケールを変えるパラメータ a による操作
  - translation: シフトパラメータ b による操作

$$x_{a,b}(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-b}{a}\right)$$

- ウェーブレットのフーリエ変換

$$X_{a,b}(\omega) = \sqrt{a} X(a\omega) \exp(-j\omega b)$$

Wavelet (2)

- Continuous Wavelet signal
  - dilation: operation by scaling parameter a
  - translation: operation by shift parameter b

$$x_{a,b}(t)=\frac{1}{\sqrt{a}}x\left(\frac{t-b}{a}\right)$$

- Fourier transform of Wavelet

$$X_{a,b}(\omega)=\sqrt{a}X(a\omega)exp(-j\omega b)$$

復習

- フーリエ変換

$$X(\omega) \equiv \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

- 信号の拡大・縮小に対するフーリエ変換

$$\int_{-\infty}^{\infty} x\left(\frac{t}{a}\right)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} x(s)\exp(-j\omega as)ads$$
$$= a\int_{-\infty}^{\infty} x(s)\exp(-j(\omega a)s)ds$$
$$= aX(a\omega)$$

$$s \equiv \frac{t}{a}$$

$$ds = \frac{1}{a}dt$$

Review

- Fourier Transform

$$X(\omega) \equiv \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

- Fourier Transform for expanded (shrunk) signal

$$\begin{aligned}\int_{-\infty}^{\infty} x\left(\frac{t}{a}\right)\exp(-j\omega t)dt &= \int_{-\infty}^{\infty} x(s)\exp(-j\omega as)ads \\ &= a\int_{-\infty}^{\infty} x(s)\exp(-j(\omega a)s)ds \\ &= aX(a\omega)\end{aligned}$$

$$s \equiv \frac{t}{a}$$

$$ds = \frac{1}{a}dt$$

復習 (2)

- 信号のシフトに対するフーリエ変換

$$\int_{-\infty}^{\infty} x(t-b)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} x(s)\exp(-j\omega(s+b))ds$$

$$= \int_{-\infty}^{\infty} x(s)\exp(-j\omega s)\exp(-j\omega b)ds$$

$$= \exp(-j\omega b)\int_{-\infty}^{\infty} x(s)\exp(-j\omega s)ds$$

$$= \exp(-j\omega b)X(\omega)$$

$$s \equiv t-b$$

$$ds = dt$$

Review (2)

- Fourier Transform for shifted signal

$$\int_{-\infty}^{\infty} x(t-b)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} x(s)\exp(-j\omega(s+b))ds$$

$$= \int_{-\infty}^{\infty} x(s)\exp(-j\omega s)\exp(-j\omega b)ds$$

$$= \exp(-j\omega b)\int_{-\infty}^{\infty} x(s)\exp(-j\omega s)ds$$

$$= \exp(-j\omega b)X(\omega)$$

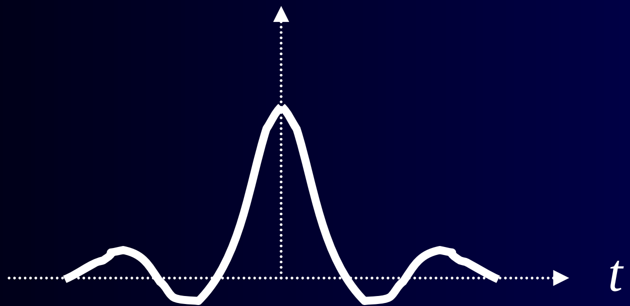
$$s \equiv t-b$$

$$ds = dt$$

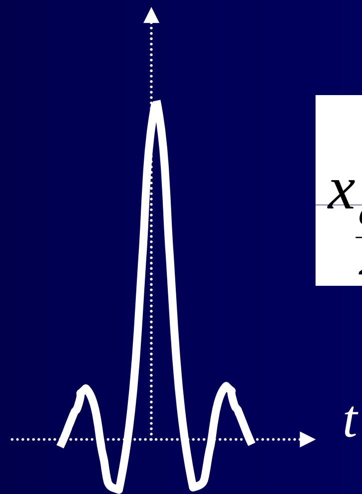
ウェーブレット (3)

- dilationによる信号波形の変化

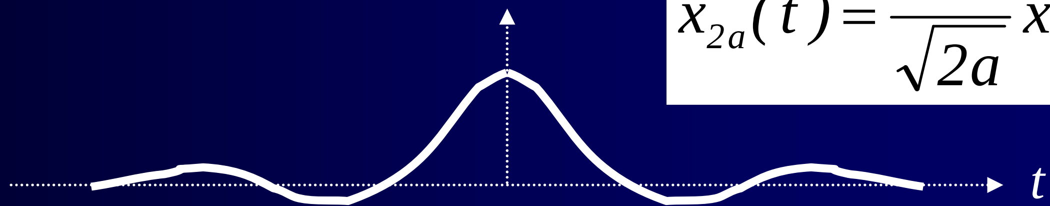
$$x_a(t)=\frac{1}{\sqrt{a}}x\left(\frac{t}{a}\right)$$



$$x_{\frac{a}{2}}(t)=\sqrt{\frac{2}{a}}x\left(\frac{2t}{a}\right)$$



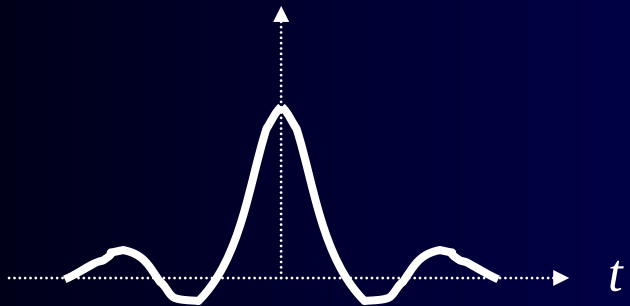
$$x_{2a}(t)=\frac{1}{\sqrt{2a}}x\left(\frac{t}{2a}\right)$$

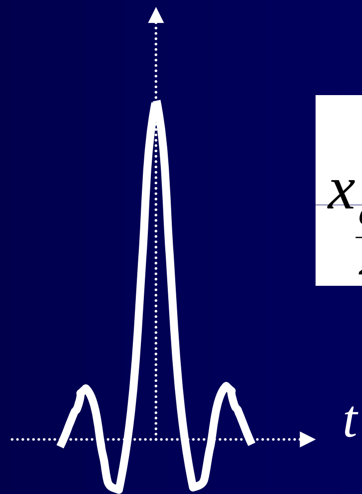


Wavelet (3)

- Signal variation by dilation

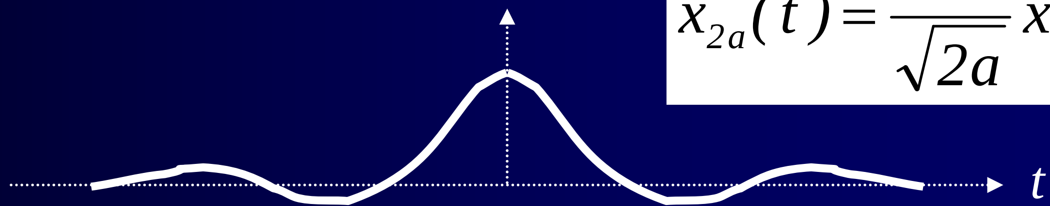
$$x_a(t) = \frac{1}{\sqrt{a}}\, x\left(\frac{t}{a}\right)$$





$$x_{\frac{a}{2}}(t) = \sqrt{\frac{2}{a}}\, x\left(\frac{2t}{a}\right)$$

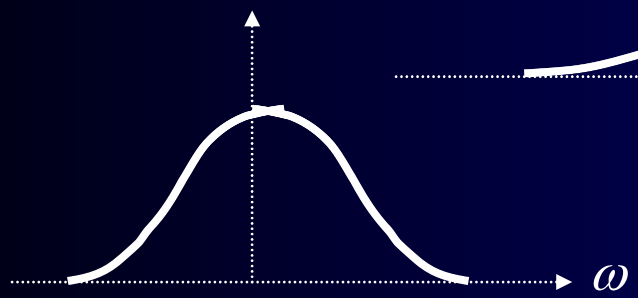
$$x_{2a}(t) = \frac{1}{\sqrt{2a}}\, x\left(\frac{t}{2a}\right)$$



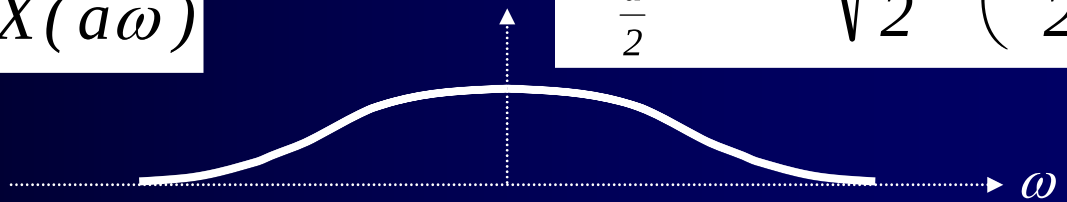
ウェーブレット (4)

- dilationによるスペクトルの変化

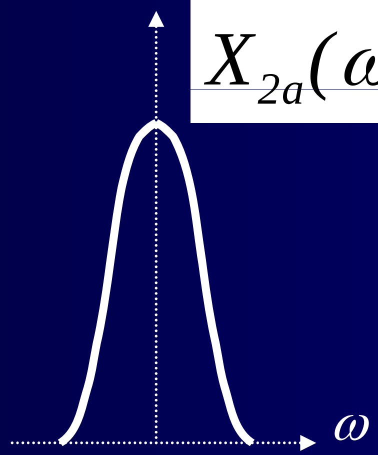
$$X_a(\omega)=\sqrt{a}X(a\omega)$$



$$X_{\frac{a}{2}}(\omega)=\sqrt{\frac{a}{2}}X\left(\frac{a\omega}{2}\right)$$



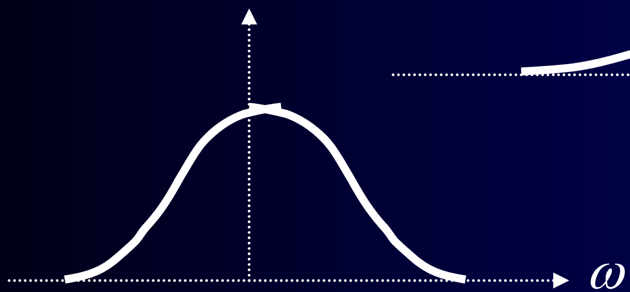
$$X_{2a}(\omega)=\sqrt{2a}X(2a\omega)$$



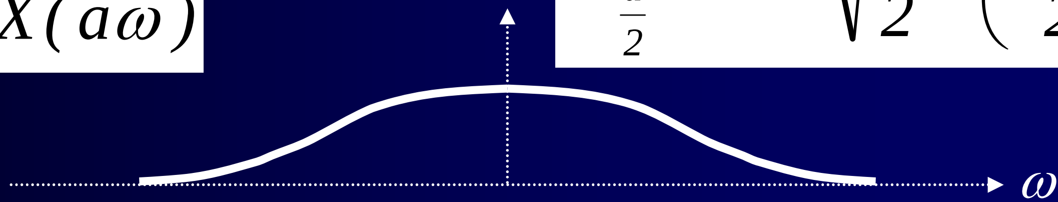
Wavelet (4)

- Spectrum variation by dilation

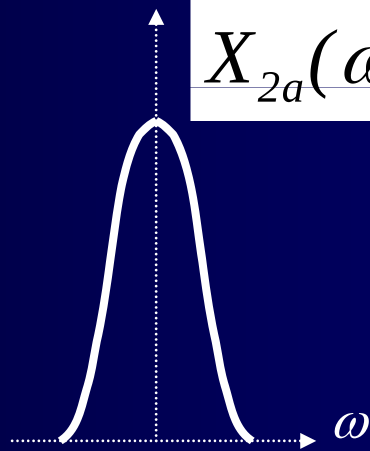
$$X_a(\omega)=\sqrt{a}X(a\omega)$$



$$X_{\frac{a}{2}}(\omega)=\sqrt{\frac{a}{2}}X\left(\frac{a\omega}{2}\right)$$

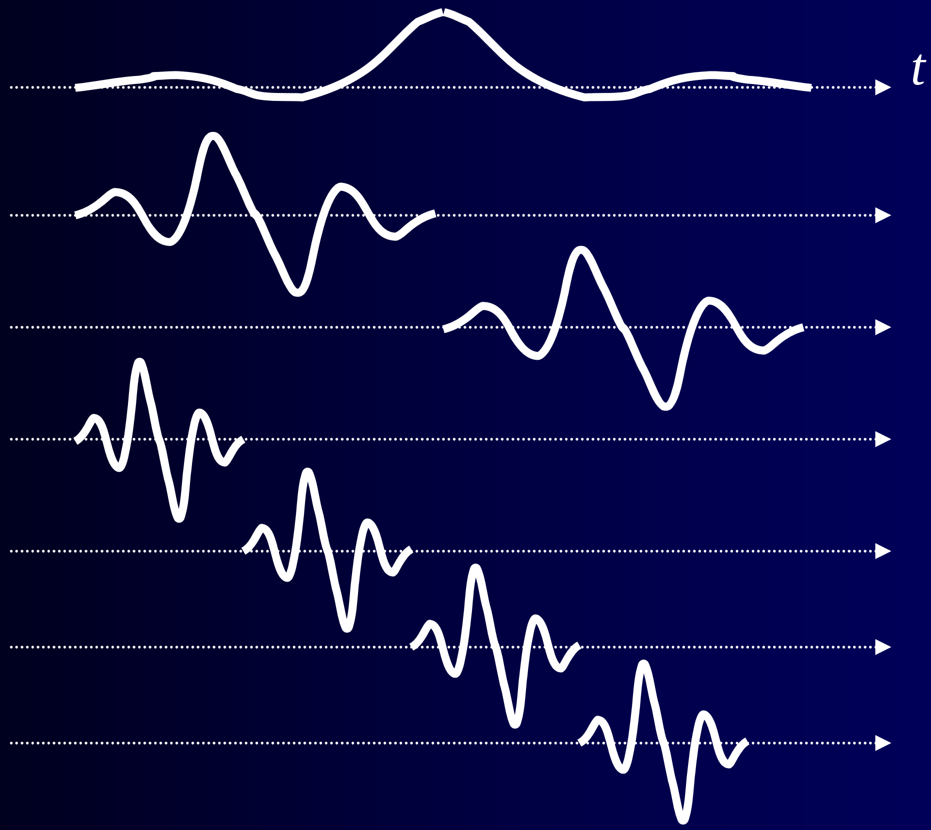


$$X_{2a}(\omega)=\sqrt{2a}X(2a\omega)$$



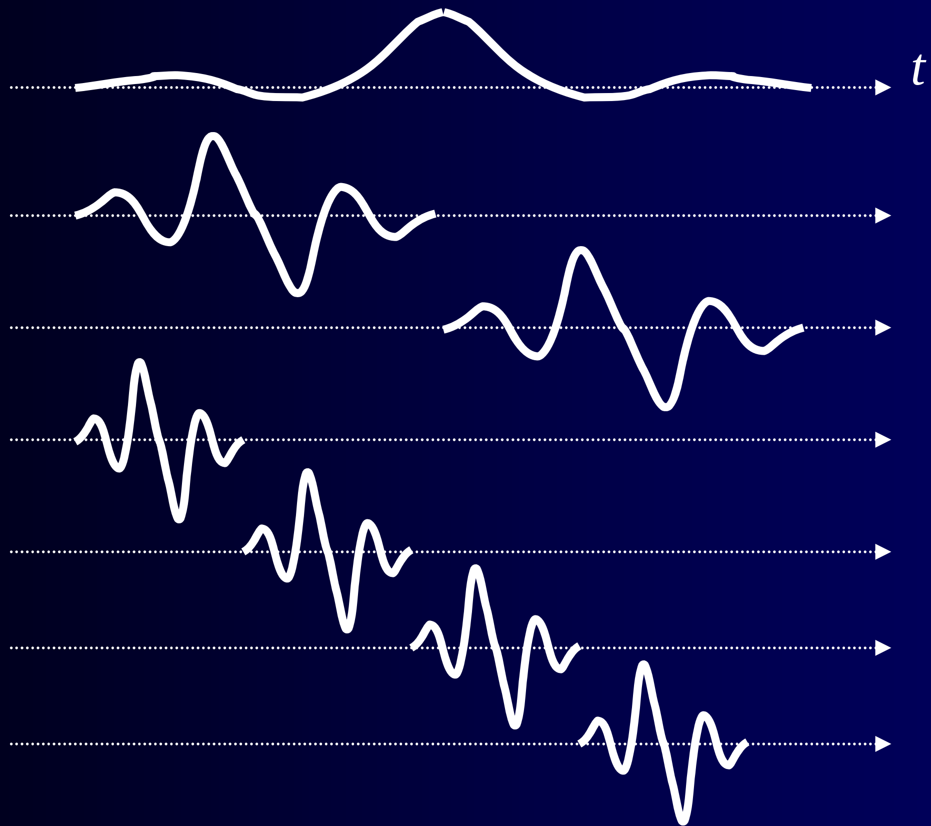
ウェーブレット変換

- Translation パラメータを用いて直交基底を作成



Wavelet Transform

- Create orthogonal basis using Translation parameter



離散ウェーブレット

- dilation と translation のパラメータを離散化

$$a = 2^m, \qquad b = n2^m$$

- 離散ウェーブレット基底関数

$$x_{m,n}(t) = 2^{-\frac{m}{2}} x(2^{-m}t - n)$$

- $x(t)$ を正規直交化した関数 $\phi(t)$ を定義すると、$H_0(z)$, $H_1(z)$ のインパルス応答$h_0(n)$, $h_1(n)$ に対して次式が成り立つ

$$\phi(t) = 2\sum_{n=0}^{N-1} h_0(n)\phi(2t-n) \qquad \varphi(t) = 2\sum_{n=0}^{N-1} h_1(n)\phi(2t-n)$$

Discrete Wavelet

- Discretized dilation and translation parameters

$$a = 2^m , \qquad b = n2^m$$

- Basis function of discrete wavelet

$$x_{m,n}(t) = 2^{-\frac{m}{2}} x(2^{-m} t - n)$$

- Define function $\phi(t)$ which is orthogonalized $x(t)$, consider $h_0(n)$, $h_1(n)$ which are impulse responses of $H_0(z)$, $H_1(z)$

$$\phi(t) = 2\sum_{n=0}^{N-1} h_0(n)\phi(2t-n) \qquad \varphi(t) = 2\sum_{n=0}^{N-1} h_1(n)\phi(2t-n)$$

ハール変換

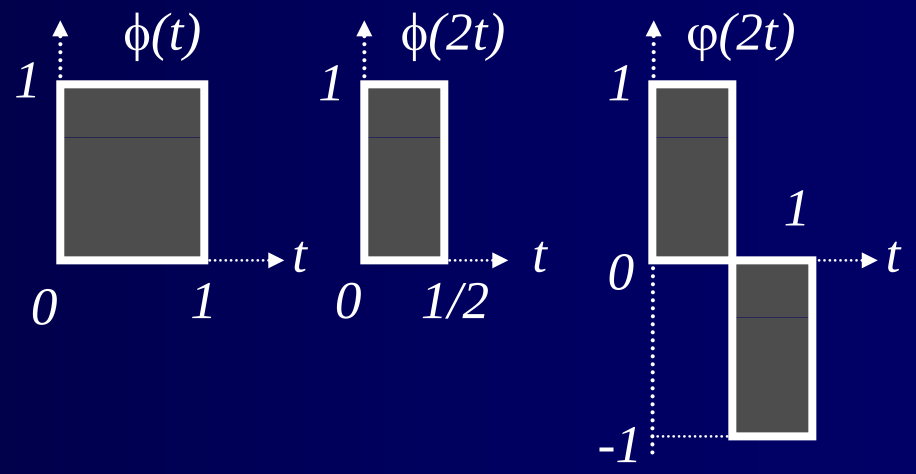
- 最も簡単なウェーブレット
- 低域フィルタ $H_0(z)$，高域フィルタ $H_1(z)$

$$H_0(z) = \frac{1+z^{-1}}{\sqrt{2}} \qquad H_1(z) = \frac{1-z^{-1}}{\sqrt{2}}$$

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & otherwise \end{cases}$$

$$\phi(t) = \phi(2t) + \phi(2t+1)$$

$$\varphi(t) = \phi(2t) - \phi(2t+1)$$



Harr Transform

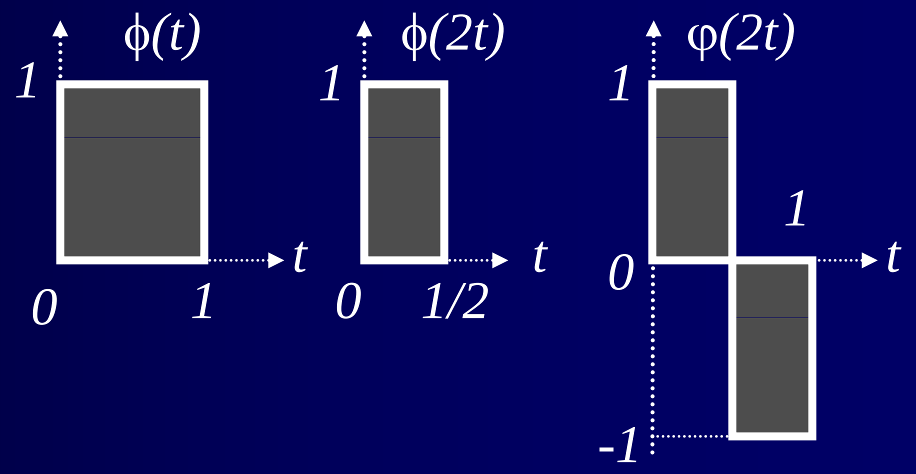
- The simplest wavelet
- LPF: $H_0(z)$, HPF: $H_1(z)$

$$H_0(z) = \frac{1 + z^{-1}}{\sqrt{2}} \qquad H_1(z) = \frac{1 - z^{-1}}{\sqrt{2}}$$

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & otherwise \end{cases}$$

$$\phi(t) = \phi(2t) + \phi(2t+1)$$

$$\varphi(t) = \phi(2t) - \phi(2t+1)$$



ハール変換 (2)

- 1次元8点ハール変換行列の例

$$
\begin{bmatrix} X(0) \\ X(1) \\ \\ \\ \vdots \\ \\ \\ X(7) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ \\ \\ \vdots \\ \\ \\ x(7) \end{bmatrix}
$$

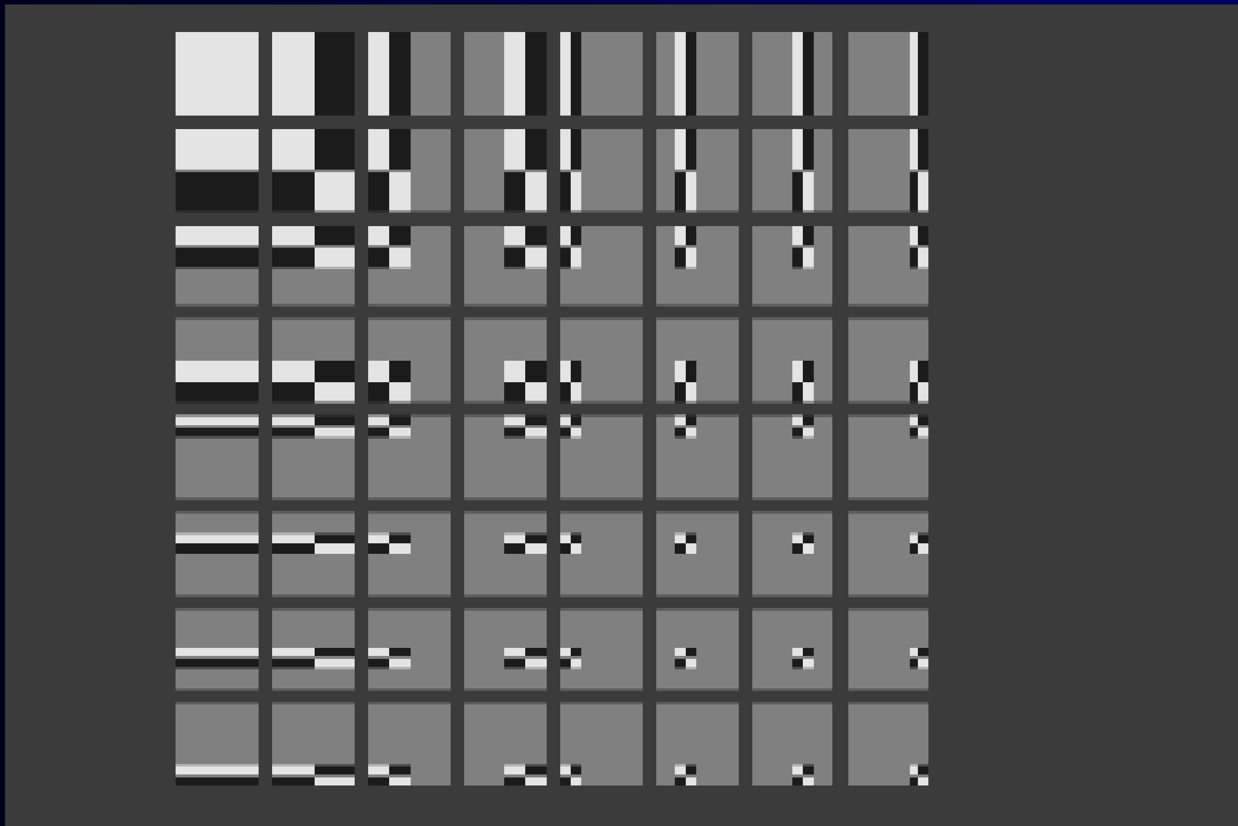
Harr Transform (2)

- 1-D 8-point Harr transform matrix

$$\begin{bmatrix} X(0) \\ X(1) \\ \\ \\ \vdots \\ \\ \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \\ \\ \vdots \\ \\ \\ x(7) \end{bmatrix}$$

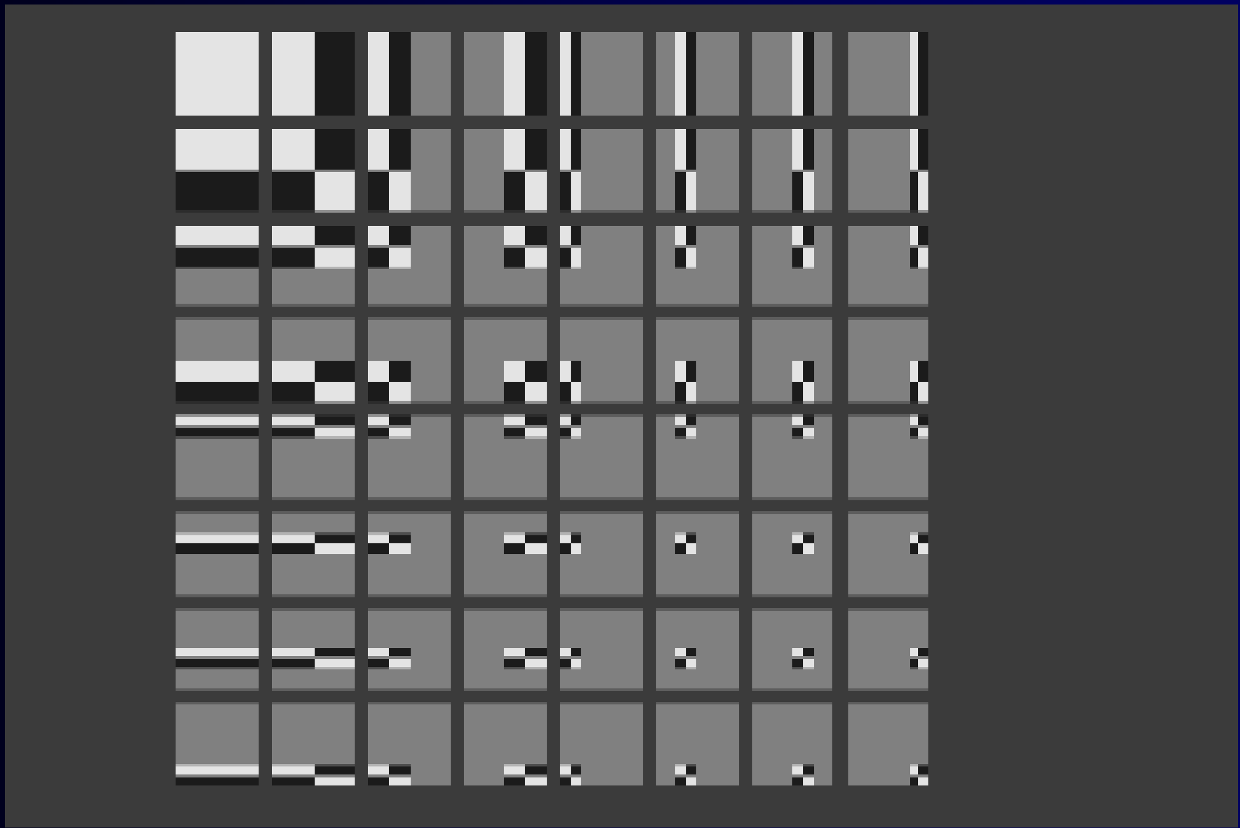
ハール変換基底

- 8x8 ハール変換の基底ベクトル



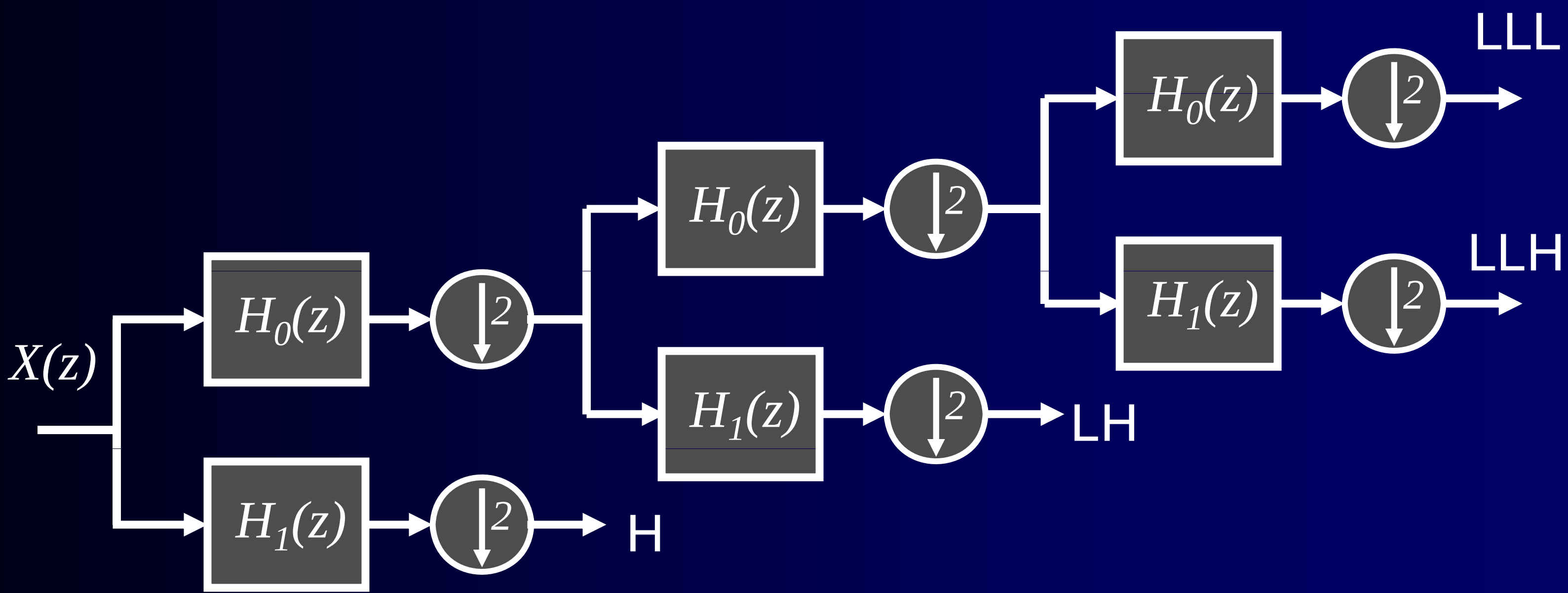
Harr Transform Basis

- 8x8 Harr transform basis



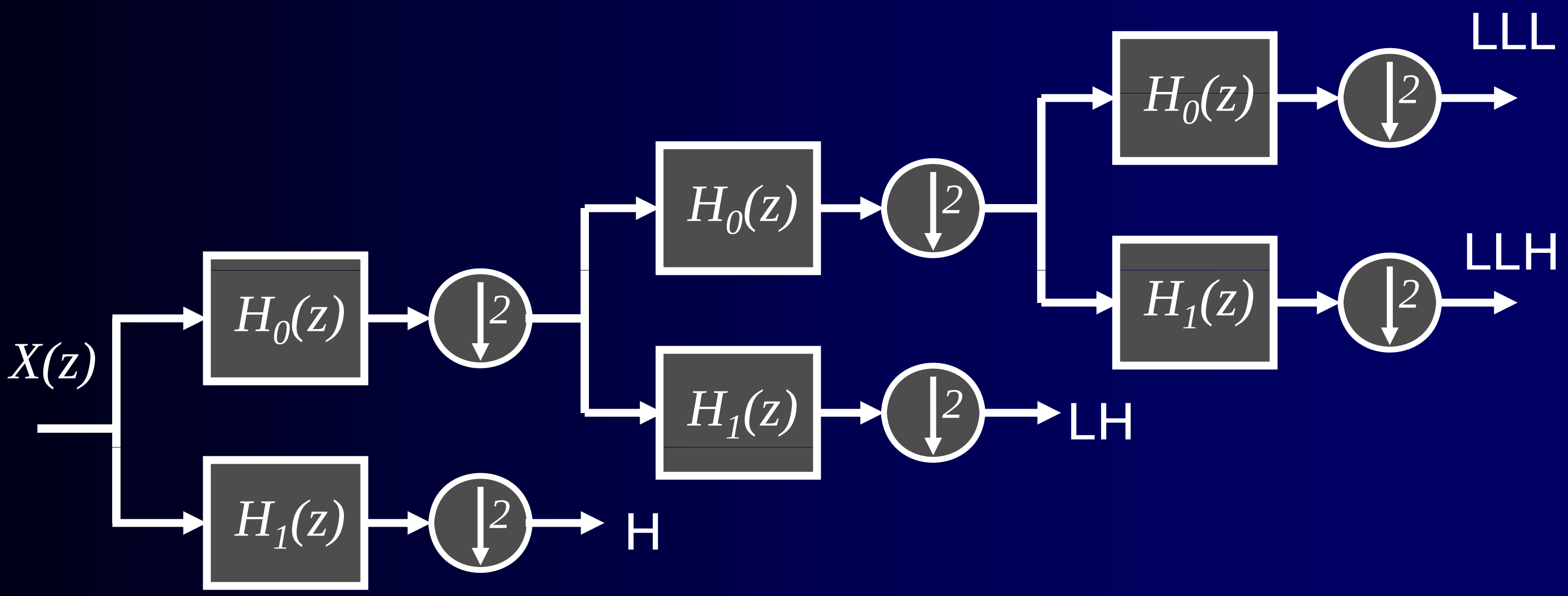
オクターブ分割

- 4バンドフィルタバンクの構成



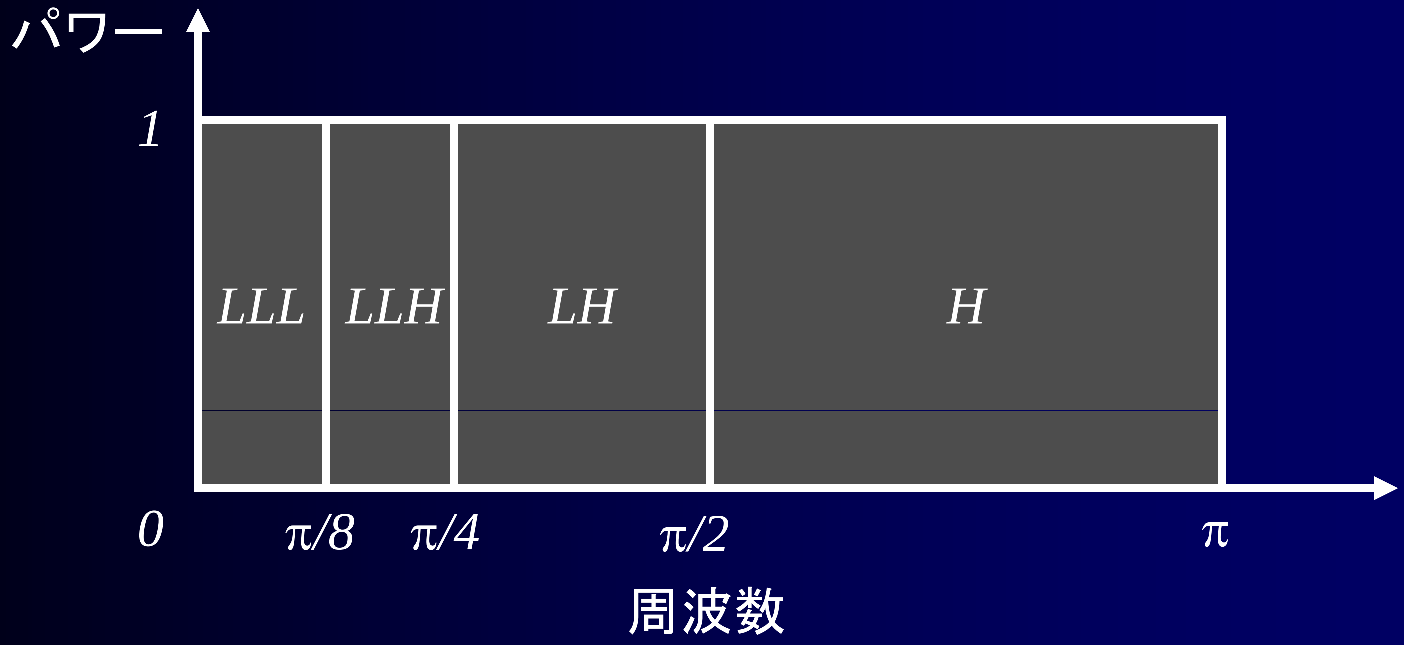
Octave Splitting

- 4-band (diadic) filterbank structure



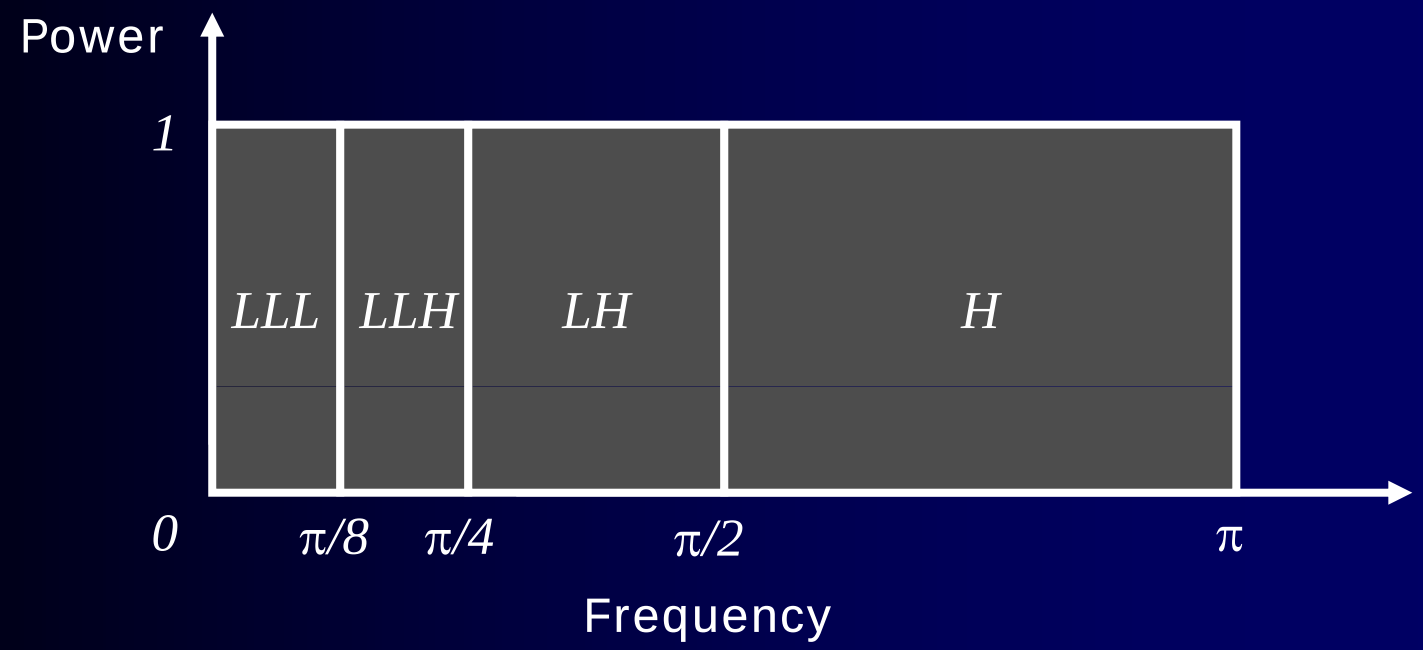
オクターブ分割 (2)

- オクターブ分割のフィルタバンクの周波数特性(4band)

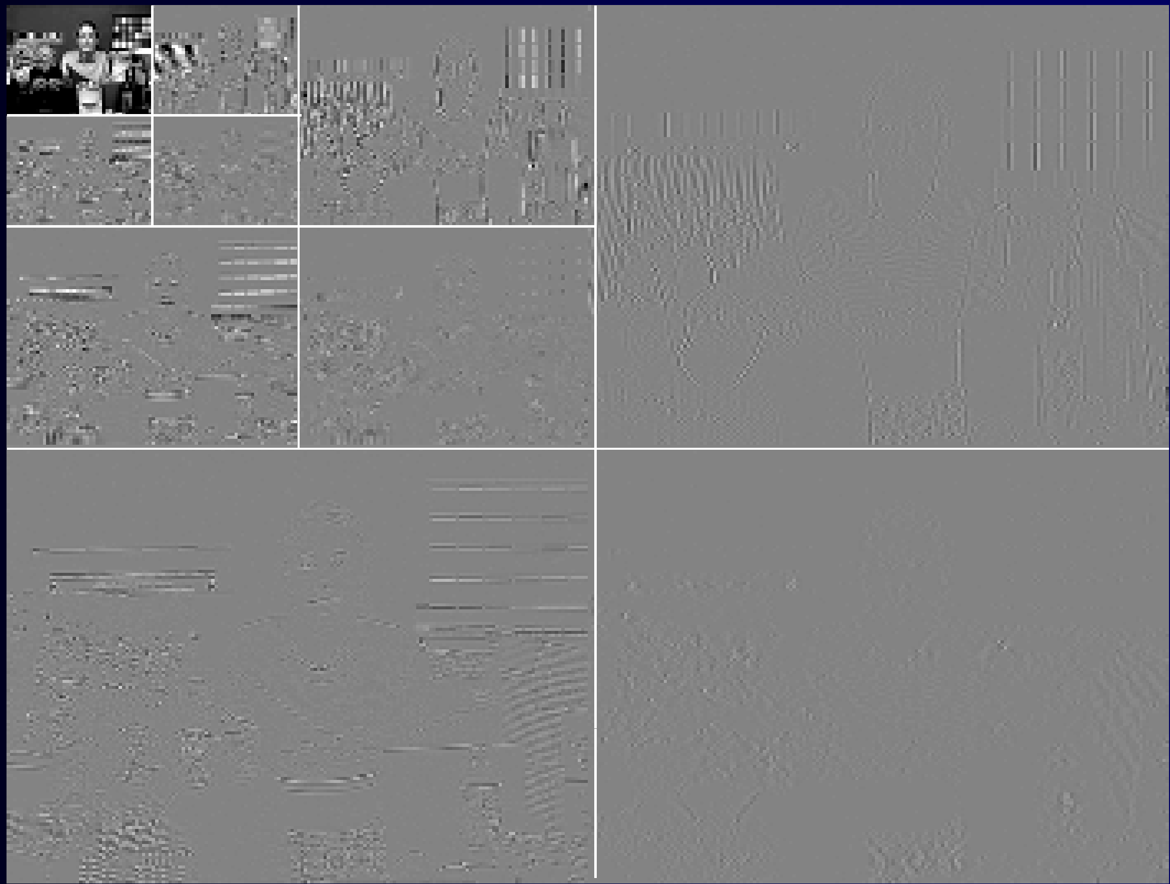


Octave Splitting (2)

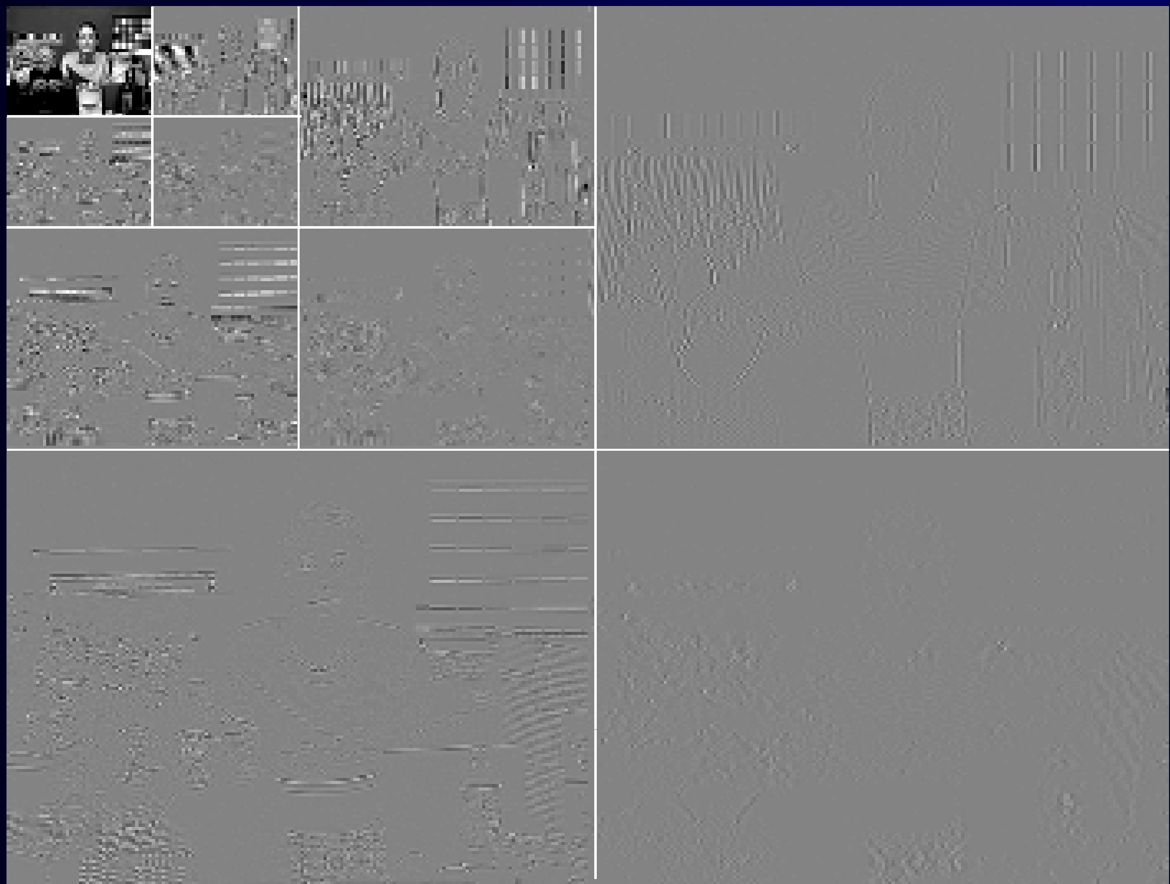
- Frequency responses of ideal octave splitting filterbank (4-band diadic)



ウェーブレット変換画像



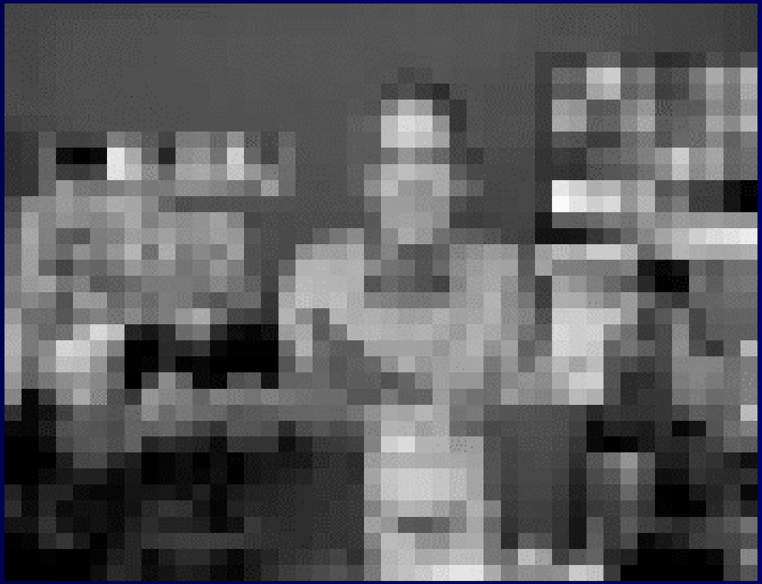
Wavelet Transform Image



ウェーブレット変換画像 (2)

- DWTとDCTの最低周波数成分(1/64)

DWT

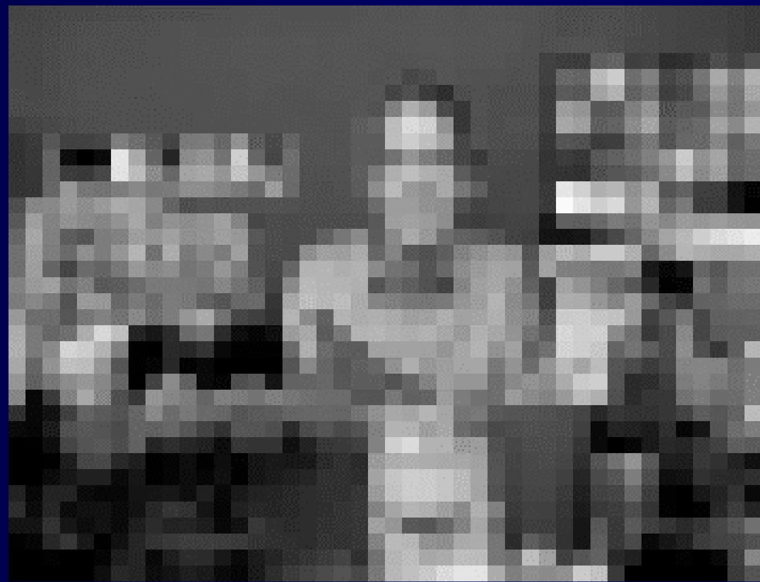


DCT

Wavelet Transform Image (2)

- The lowest frequency component of DWT and DCT (1/64)

DWT



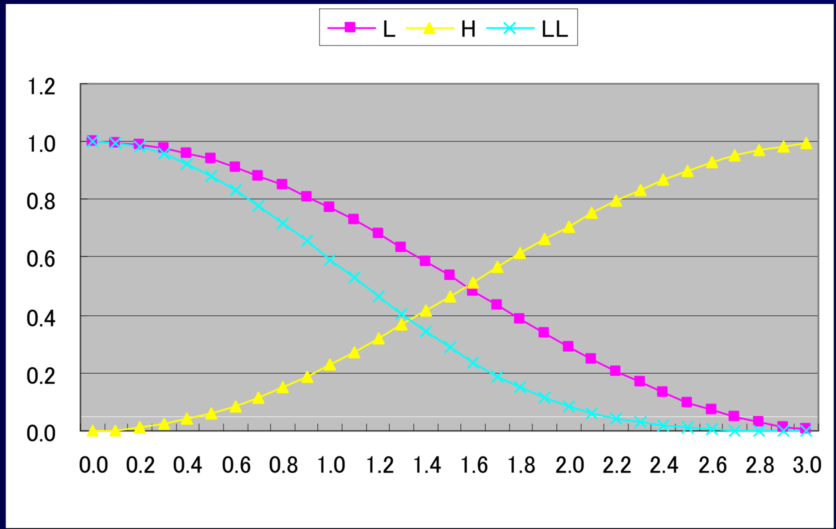
DCT

周波数特性の計算例

- Excel による周波数特性の計算

$$H_0(z) = \frac{1}{4}z^{-1} + \frac{1}{2} + \frac{1}{4}z^{1}$$

$$H_1(z) = -\frac{1}{4}z^{-1} + \frac{1}{2} + -\frac{1}{4}z^{1}$$

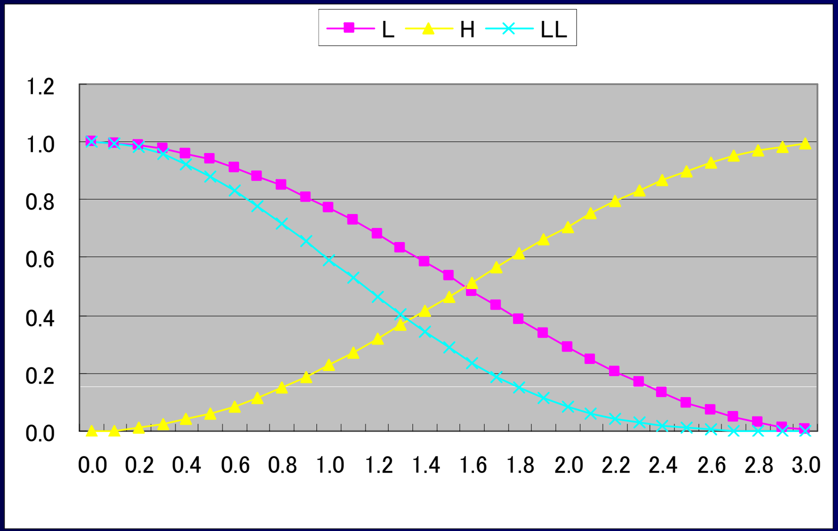


Example of Frequency Response

- Calculation of frequency response using Excel

$$H_0(z) = \frac{1}{4}z^{-1} + \frac{1}{2} + \frac{1}{4}z^{1}$$

$$H_1(z) = -\frac{1}{4}z^{-1} + \frac{1}{2} + -\frac{1}{4}z^{1}$$



JPEG 2000

- Joint Photographic Experts Group
  - ISO/IEC JTC1/SC29/WG1 で作成
  - 静止画像(RGB各8ビットの24ビットカラー画像)
  - 非可逆から可逆まで連続的に導出可能な符号化
  - ウェーブレット +EBCOT(算術符号化)
  - ISO/IEC 15444: 2000

- ウェーブレット
  - リフティング，線形量子化
  - EBCOT(Embedded Block Coding with Optimized Truncation), 複数コンテクストの算術符号化(MQ-coder)

JPEG 2000

- Joint Photographic Experts Group
  - ISO/IEC JTC1/SC29/WG1
  - Still Image(RGB各8ビットの24ビットカラー画像)
  - From lossy to lossless
  - Wavelet +EBCOT(Arithmetic coding)
  - ISO/IEC 15444: 2000

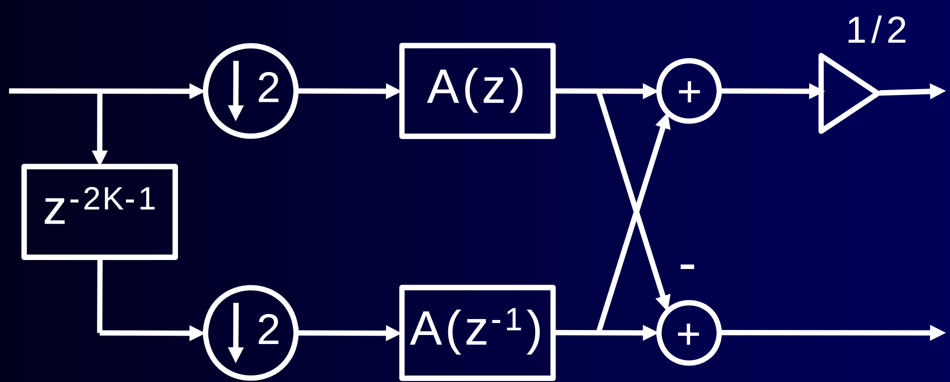
- Wavelet
  - Lifting scheme, linear quantizer
  - EBCOT(Embedded Block Coding with Optimized Truncation), multi-context arithmetic coding(MQ-coder)

オールパスフィルタによる実現

- オールパスフィルタA(z)によるLPFとHPF

$$H_0(z) = \frac{\sqrt{2}}{2}\left\{z^{-2K-1}A(z^{-2}) + A(z^2)\right\}$$

$$H_1(z) = \frac{\sqrt{2}}{2}\left\{z^{-2K-1}A(z^{-2}) - A(z^2)\right\}$$

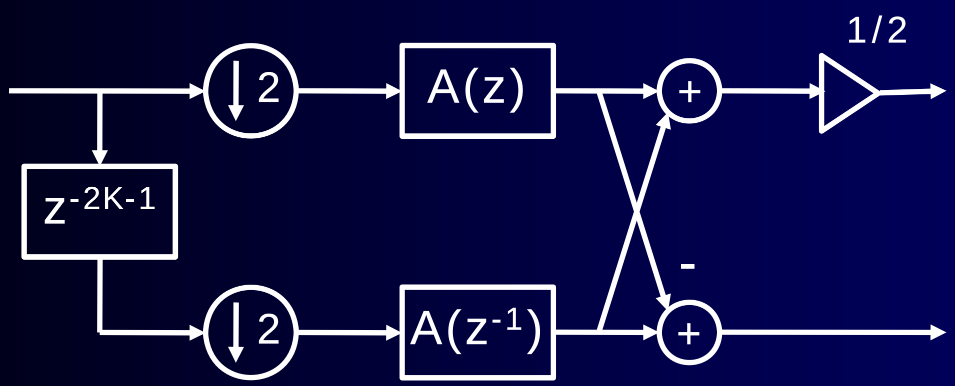


Representation by All Pass Filter

- LPF, HPF are created by all pass filter A(z)

$$H_0(z) = \frac{\sqrt{2}}{2}\left\{z^{-2K-1}A(z^{-2}) + A(z^2)\right\}$$

$$H_1(z) = \frac{\sqrt{2}}{2}\left\{z^{-2K-1}A(z^{-2}) - A(z^2)\right\}$$



オールパスフィルタ

- 次数N次のオールパスフィルタ

$$A(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}}$$

- 周波数特性

$$A(e^{j\theta}) = z^{-jN\omega} \frac{\sum_{n=0}^{N} a_n \{\cos(n\omega) + j\sin(n\omega)\}}{\sum_{n=0}^{N} a_n \{\cos(n\omega) - j\sin(n\omega)\}}$$

$$\theta(\omega) = -N\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin(n\omega)}{\sum_{n=0}^{N} a_n \cos(n\omega)}$$

All Pass Filter

- N-th order all pass filter A(z)

$$A(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}}$$

- Frequency responses

$$A(e^{j\theta}) = z^{-jN\omega} \frac{\sum_{n=0}^{N} a_n \{\cos(n\omega) + j\sin(n\omega)\}}{\sum_{n=0}^{N} a_n \{\cos(n\omega) - j\sin(n\omega)\}}$$

$$\theta(\omega) = -N\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin(n\omega)}{\sum_{n=0}^{N} a_n \cos(n\omega)}$$

オールパスフィルタ (2)

- LPF，HPFの周波数特性は直線位相

$$H_0(e^{j\omega}) = \sqrt{2}^{-j(k+\frac{1}{2})} \cos\left[\theta(2\omega) + (k+\frac{1}{2})\omega\right]$$

$$H_1(e^{j\omega}) = \sqrt{2}^{-j(k+\frac{1}{2})} \sin\left[\theta(2\omega) + (k+\frac{1}{2})\omega\right]$$

- 振幅は直交

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 2$$

All Pass Filter (2)

- Frequency Characteristics of LPF, HPF are linear phase

$$H_0(e^{j\omega}) = \sqrt{2}^{-j(k+\frac{1}{2})} \cos\left[\theta(2\omega) + (k+\frac{1}{2})\omega\right]$$

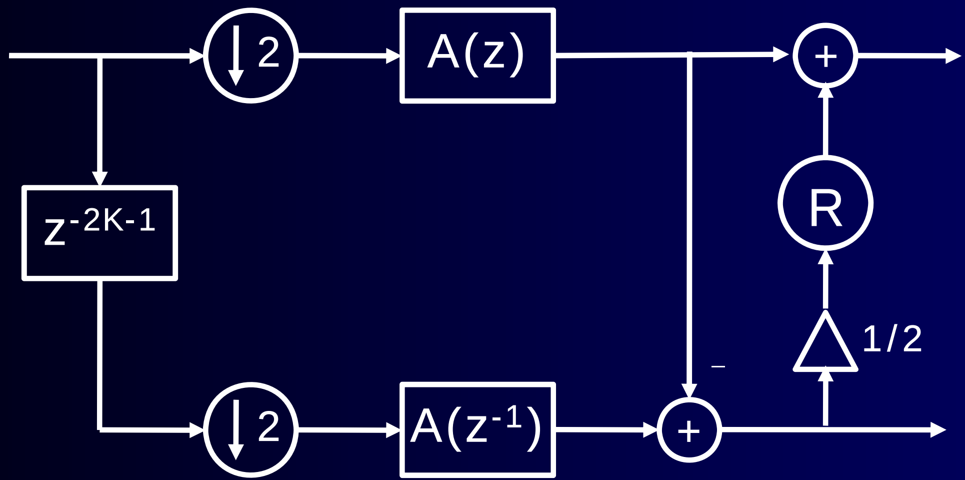
$$H_1(e^{j\omega}) = \sqrt{2}^{-j(k+\frac{1}{2})} \sin\left[\theta(2\omega) + (k+\frac{1}{2})\omega\right]$$

- Amplitudes are in an orthogonal relation

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 2$$

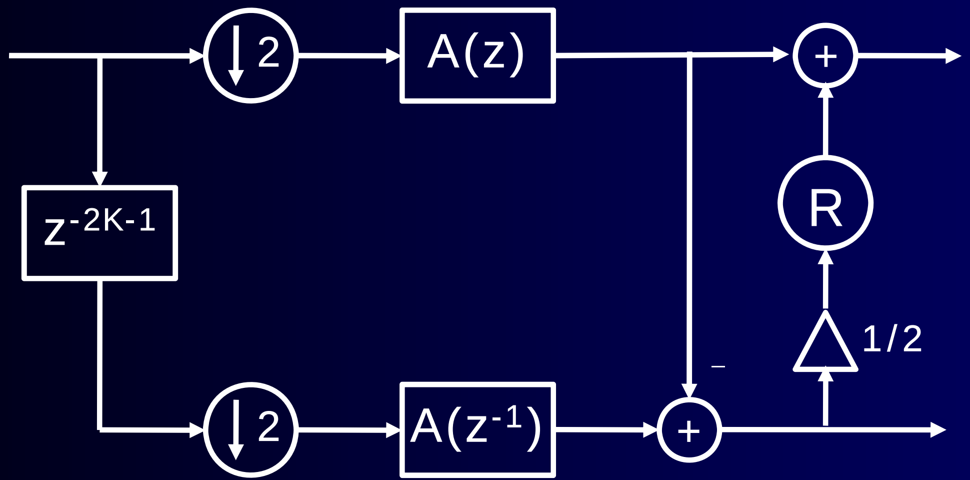
リフティングによる実現

- リフティング(R)による整数化処理



Representation by Lifting

- Integer operation by (R)



問題

- S変換(y(1)=(a+b)/2, y(2)=a-b)を以下の2次元データに適用せよ.

128	132	134	141
126	128	130	135
120	125	128	133
125	130	134	131

Quiz

- Apply S-transform ($y(1)=(a+b)/2$, $y(2)=a-b$) to the 2-dimensional data below.

128	132	134	141
126	128	130	135
120	125	128	133
125	130	134	131