信号理論

- No.7 サブバンドと完全再構成 -

渡辺　裕

Signal Theory

- No.7 Subband and Perfect Reconstruction -

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サブバンド

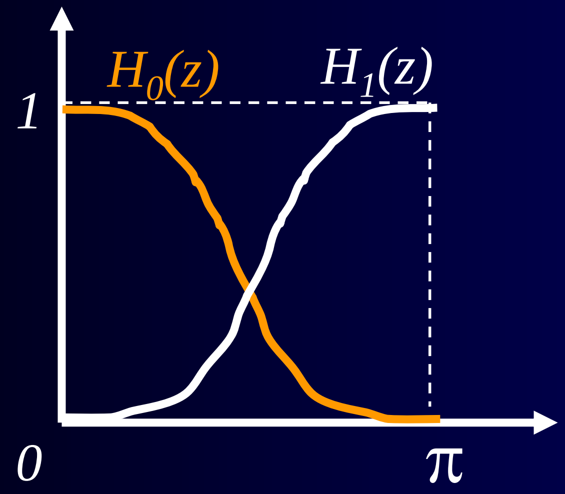
- 通過周波数帯域の限られたフィルタを適用することにより，特定の周波数帯域のサンプルだけを処理する手法
- 用いるフィルタをサブバンドフィルタと呼ぶ
- サブバンド出力を間引き、量子化することにより、変換符号化と等価になる
- チャネルイコライザー（伝送等化器）、オーディオイコライザー（オーディオ周波数調整器）、オーディオ符号化などに応用される
- 主にFIRフィルタが用いられる

Subband

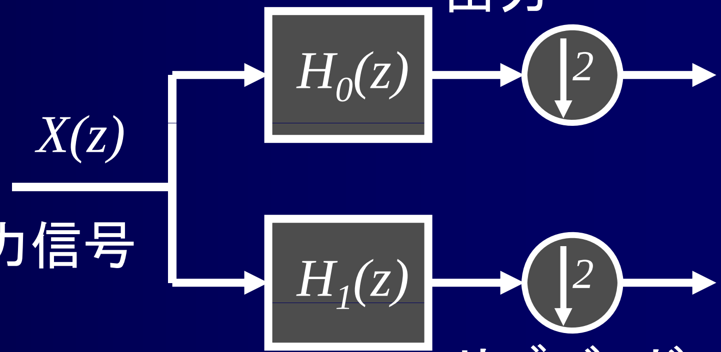
- Processing samples having specified frequency range by applying limited bandwidth filter
- Filter is called “Subband Filter”
- It equals to transform coding by applying decimation and quantization to subband outputs
- Application: Channel equalizer (Transmission equalizer), Audio equalizer, Audio coding
- In most cases, FIR filter

サブバンドフィルタ

- 最も簡単なサブバンド処理は2バンド分割
- 低域通過型FIRフィルタ $H_0(z)$ および高域通過型FIRフィルタ $H_1(z)$ による構成
- サブバンド出力は1:2にダウンサンプル（サブサンプル）される

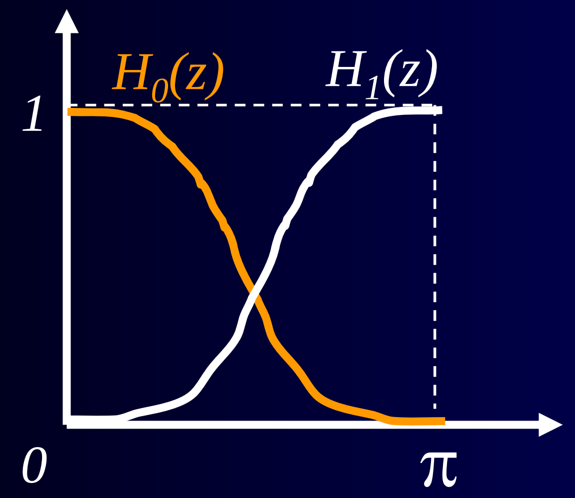


フィルタの周波数特性

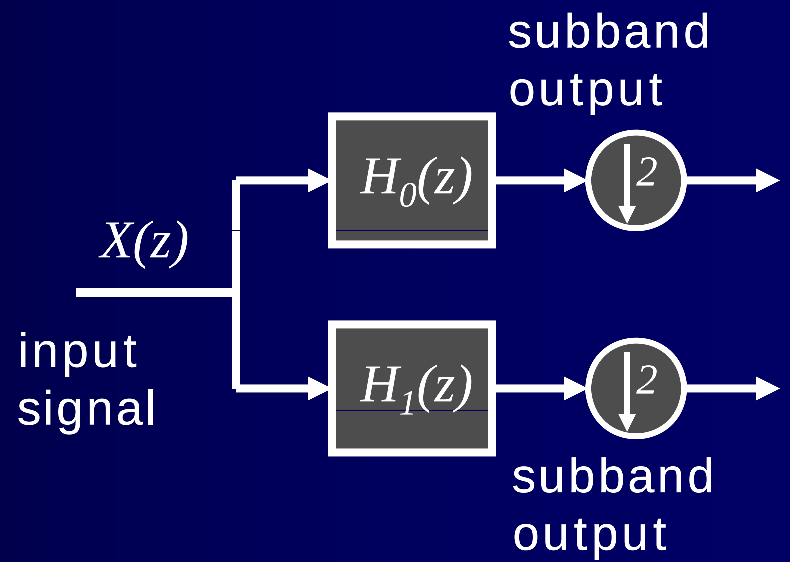


Subband Filter

- The simplest subband is two-band
- Consists of Low pass FIR filter $H_0(z)$ and High pass FIR filter $H_1(z)$
- Subband output is down-sampled to 1:2 (subsample)

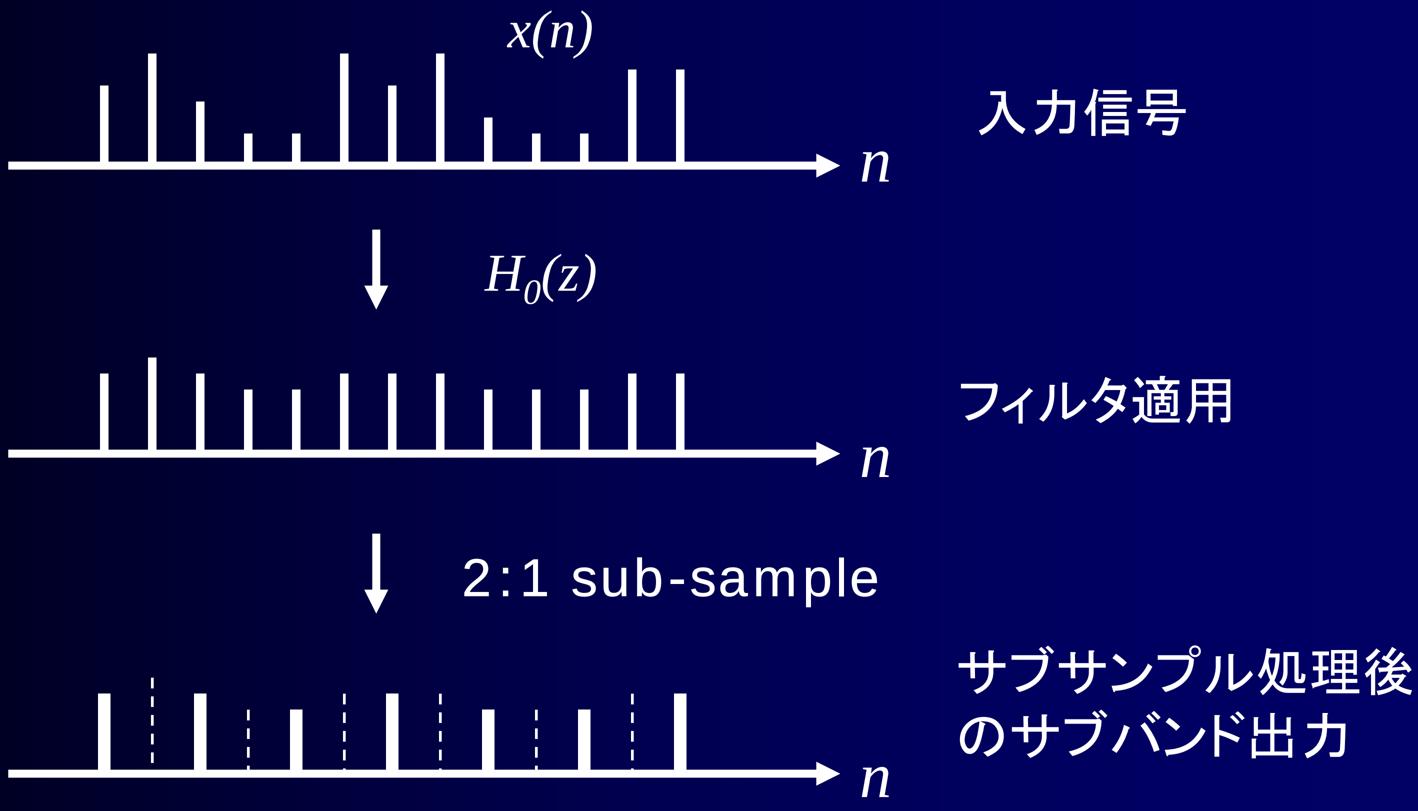


Filter's frequency response



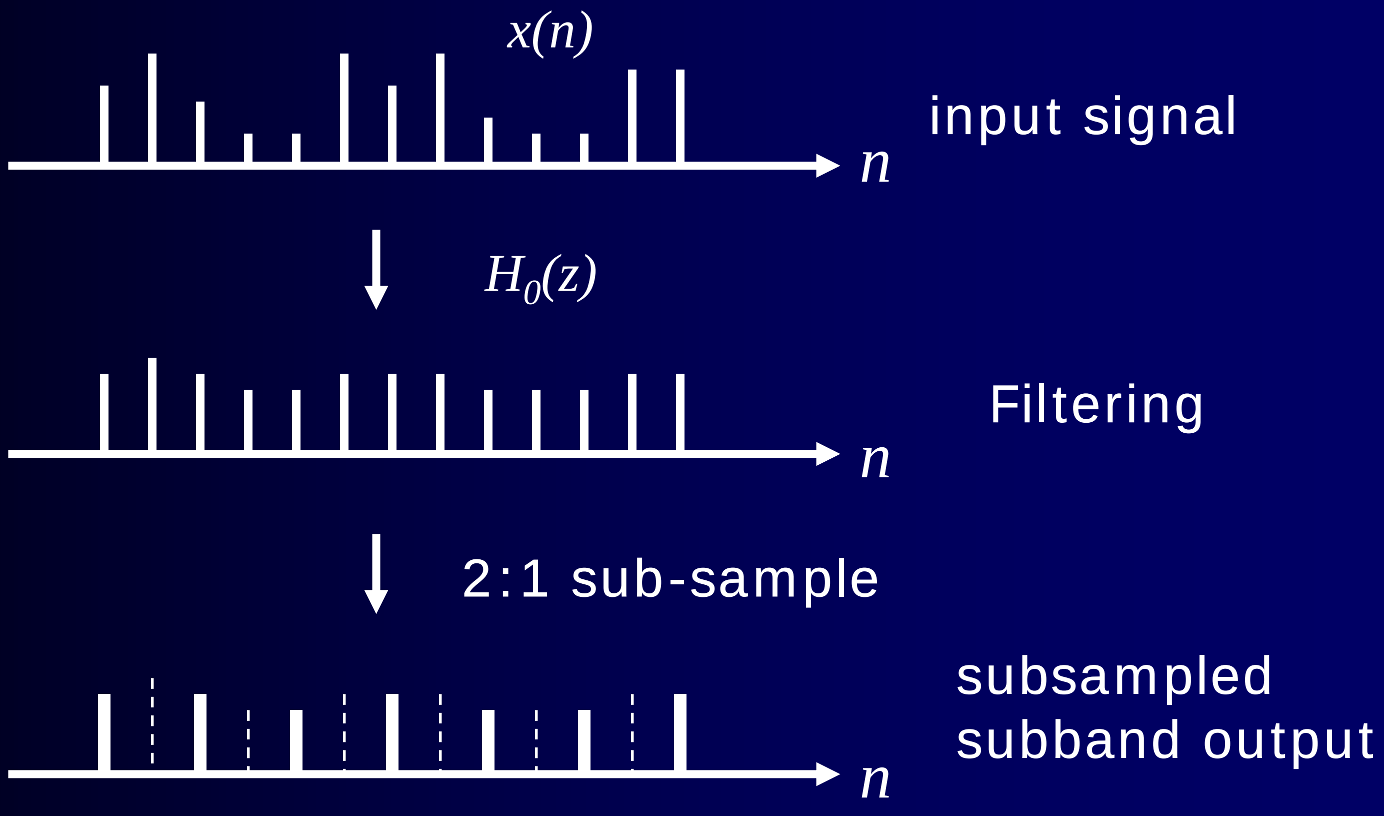
サブバンドフィルタ (2)

- 2:1サブサンプル



Subband Filter (2)

- 2:1Subsample

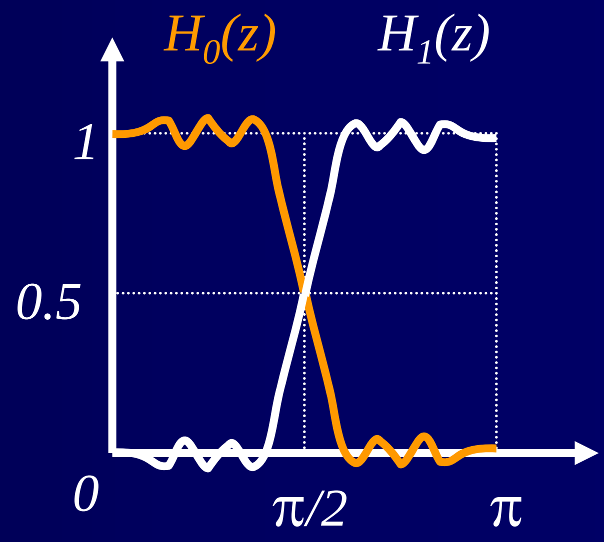


ハーフバンドフィルタ

- ハーフバンドフィルタ H(z) の定義

$$H(z) = H(z^{-1})$$

$$H(z) + H(-z^{-1}) = 1$$



- 周波数特性
  - 周波数特性は π/2 で対称
  - 対称なフィルタ特性と加算すると1になる

$$H(e^{j\omega}) + H(-e^{-j\omega}) = H(e^{j\omega}) + H(e^{j(\pi-\omega)}) = 1$$

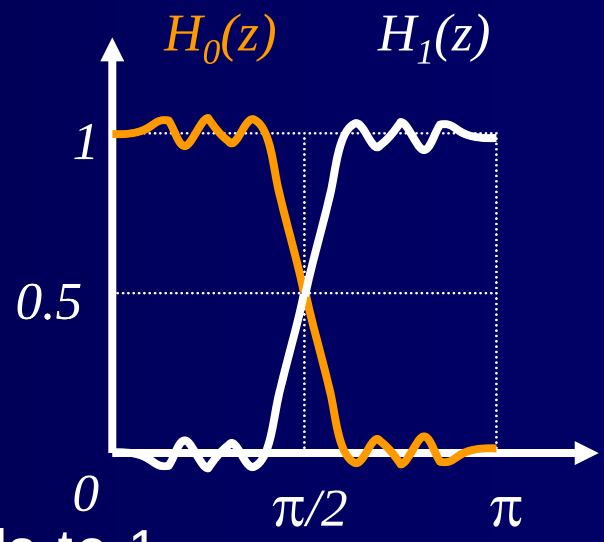
Halfband Filter

- Definition of halfband filter H(z)

$$H(z) = H(z^{-1})$$

$$H(z) + H(-z^{-1}) = 1$$

- Frequency response
  - Symmetric at $\pi/2$
  - Sum of two filters response equals to 1



$$H(e^{j\omega}) + H(-e^{-j\omega}) = H(e^{j\omega}) + H(e^{j(\pi-\omega)}) = 1$$

ハーフバンドフィルタ (2)

- インパルス応答の性質

$$h(n)+(-1)^n h(n)=\delta(n)$$

$$h(n)=\begin{cases} 1/2 & (n=0) \\ 0 & (n=even, n\neq 0) \end{cases}$$

$$H(z)=\sum_{n=-\infty}^{\infty} h(n)z^{-n}=\cdots+h(-1)z^{1}+h(0)z^{0}+h(1)z^{-1}+\cdots$$

$$H(z^{-1})=\sum_{n=-\infty}^{\infty} h(n)z^{n}=\cdots+h(-1)z^{-1}+h(0)z^{0}+h(1)z^{1}+\cdots$$

$$H(-z^{-1})=\sum_{n=-\infty}^{\infty} h(n)(-z)^{n}=\cdots h(-2)z^{-2}-h(-1)z^{-1}+h(0)z^{0}$$
$$-h(1)z^{1}+h(2)z^{2}\cdots$$

Halfband Filter (2)

- Characteristic of impulse response

$$h(n)+(-1)^n h(n)=\delta(n)$$

$$h(n)=\begin{cases} 1/2 & (n=0) \\ 0 & (n=even, n\neq 0) \end{cases}$$

$$H(z)=\sum_{n=-\infty}^{\infty} h(n)z^{-n}=\cdots+h(-1)z^{1}+h(0)z^{0}+h(1)z^{-1}+\cdots$$

$$H(z^{-1})=\sum_{n=-\infty}^{\infty} h(n)z^{n}=\cdots+h(-1)z^{-1}+h(0)z^{0}+h(1)z^{1}+\cdots$$

$$H(-z^{-1})=\sum_{n=-\infty}^{\infty} h(n)(-z)^{n}=\cdots h(-2)z^{-2}-h(-1)z^{-1}+h(0)z^{0}$$

$$-h(1)z^{1}+h(2)z^{2}\cdots$$

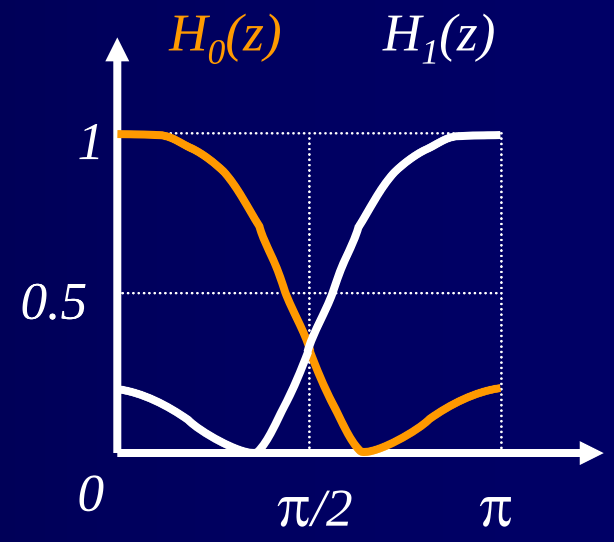
ミラーフィルタ

- ミラーフィルタの定義

$$H_1(z) = H_0(-z)$$

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$

$$h_1(n) = (-1)^n h_0(n)$$

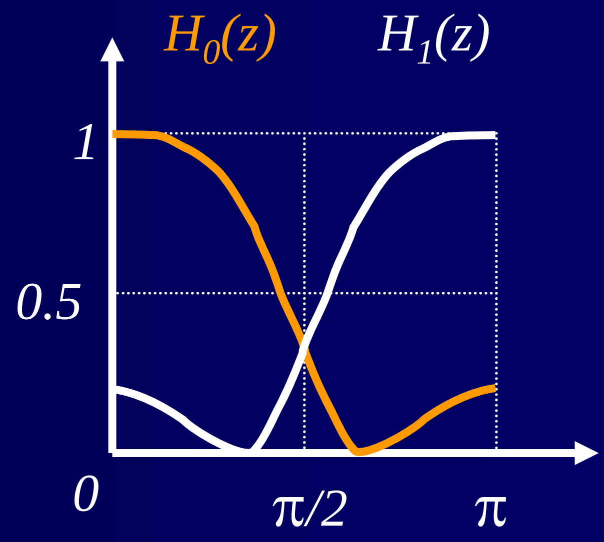


- 周波数特性
  - 周波数特性は π/2 で対称
  - 対称なフィルタ特性と加算すると1にならない

Mirror Filter

- Definition of mirror filter

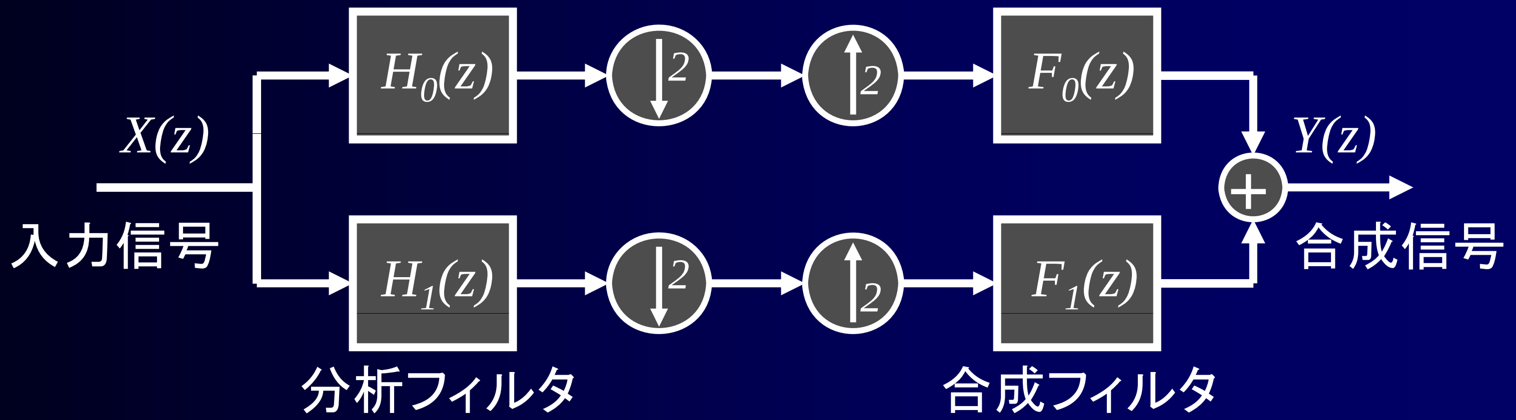
$$H_1(z) = H_0(-z)$$
$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$h_1(n) = (-1)^n h_0(n)$$



- Frequency response
  - Symmetric at $\pi/2$
  - Sum of two filters response does not equals to 1

分析･合成

- 分析フィルタ部と合成フィルタ部

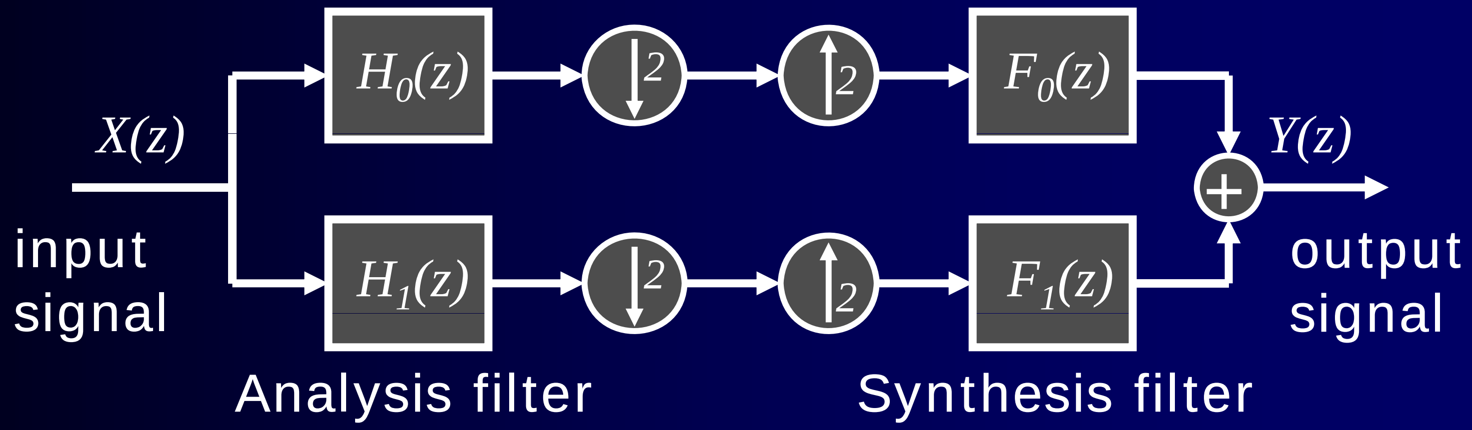


- 完全再構成の条件

$$H_0(-z)F_0(z)+H_1(-z)F_1(z)=0$$
$$H_0(z)F_0(z)+H_1(z)F_1(z)=2z^{-L}$$

Analysis and Synthesis

- Analysis filters and Synthesis filters



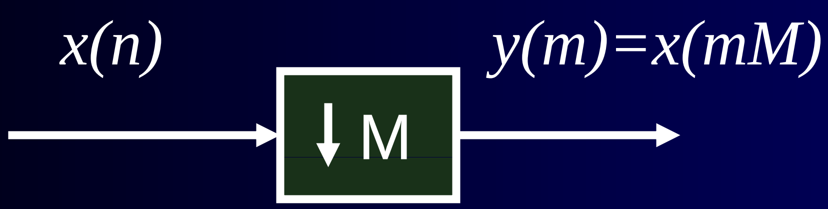
- Condition for Perfect Reconstruction

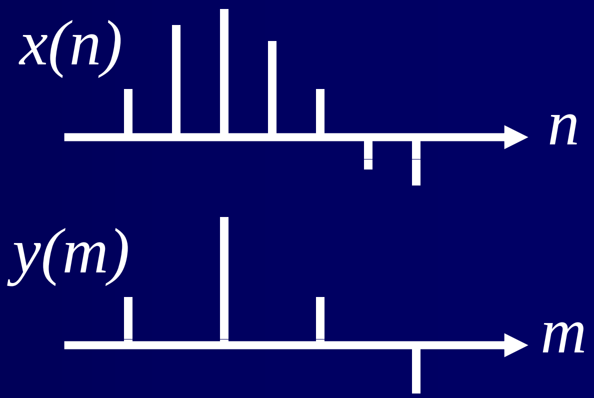
$$H_0(-z)F_0(z)+H_1(-z)F_1(z)=0$$
$$H_0(z)F_0(z)+H_1(z)F_1(z)=2z^{-L}$$

完全再構成

- ダウンサンプルのz変換表現

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} \exp(-j2\pi k / M)\right)$$

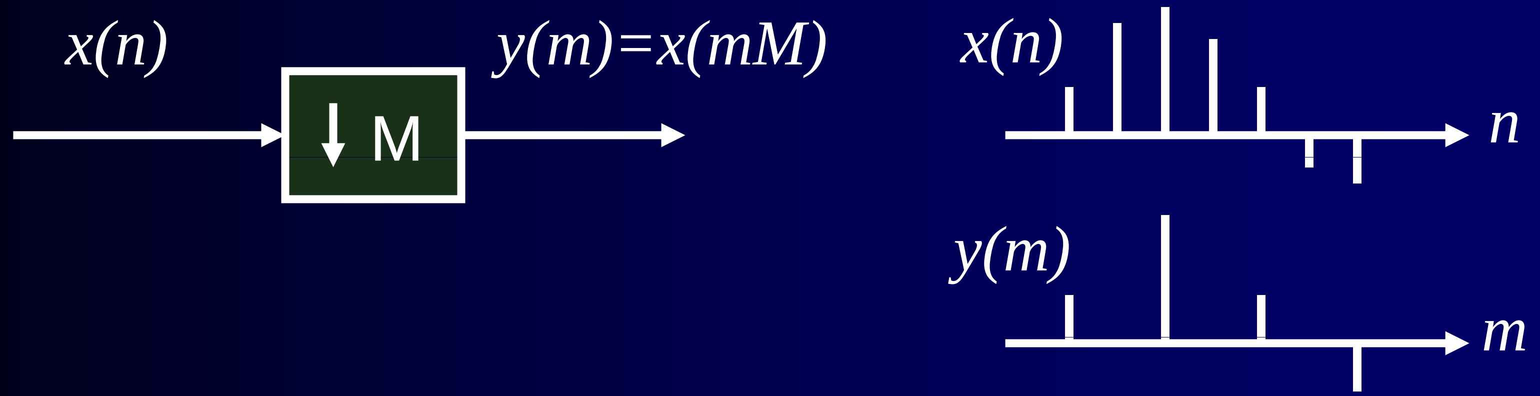




Perfect Reconstruction

- Z-transform representation for down sampling

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} \exp(-j2\pi k/M)\right)$$



完全再構成 (2)

- ダウンサンプリングはM周期デルタ関数と $x(n)$ の畳み込み

$$i(n) = \sum_{r=-\infty}^{\infty} \delta(n - rM) = \frac{1}{M} \sum_{k=0}^{M-1} exp(j2\pi nk / M)$$

- デルタ関数は $n=\ldots-2M, -M, 0, M, 2M\ldots$ で 1, 他の場合には 0
- 上式右辺は, 例えば $M=3$ のとき, $n=3k$ (k:整数) で 1, 他の場合 0

$$i(n) = \frac{1}{3}\left(1 + e^{j\pi n2/3} + e^{j\pi n4/3}\right)$$

Perfect Reconstruction (2)

- Down sampling is convolution of M period delta function and $x(n)$

$$i(n) = \sum_{r=-\infty}^{\infty} \delta(n - rM) = \frac{1}{M} \sum_{k=0}^{M-1} exp(j2\pi nk / M)$$

- delta function is 1 at $n=\ldots-2M, -M, 0, M, 2M\ldots$, otherwise 0
- Right side of above equation equals to 1 at $n=3k$ (k:integer) for $M=3$, otherwise 0

$$i(n) = \frac{1}{3}\left(1 + e^{j\pi n 2/3} + e^{j\pi n 4/3}\right)$$

完全再構成 (3)

- $i(n)$ と $x(n)$ の積は

$$i(n)x(n) = \frac{1}{M}\sum_{k=0}^{M-1} x(n)\exp(-j2\pi nk/M)$$

- この式をz変換すれば

$$\sum_{n=-\infty}^{\infty} \frac{1}{M}\sum_{k=0}^{M-1} x(n)\exp(-j2\pi nk/M)z^{-n}$$

$$= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x(n)\exp(-jn(\omega - j2\pi k/M))$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X(z\exp(-j2\pi k/M))$$

Perfect Reconstruction (3)

- Product of *i(n)* and *x(n)*

$$i(n)x(n) = \frac{1}{M}\sum_{k=0}^{M-1} x(n)\exp(-j2\pi nk/M)$$

- By applying z-transform

$$\sum_{n=-\infty}^{\infty} \frac{1}{M}\sum_{k=0}^{M-1} x(n)\exp(-j2\pi nk/M)z^{-n}$$

$$= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x(n)\exp(-jn(\omega - j2\pi k/M))$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X(z\exp(-j2\pi k/M))$$

復習

- z変換

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (z \equiv e^{j\omega})$$

Review

- z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (z \equiv e^{j\omega})$$

完全再構成 (4)

- 1/Mへの間引きは，z変換では以下のように表される

$$\sum_{n=-\infty}^{\infty} x(nM)z^{-n} = \sum_{s=-\infty}^{\infty} x(s)z^{\frac{1}{M}(-s)} = X(z^{\frac{1}{M}})$$

$$s \equiv nM$$

$$n = \frac{1}{M}s$$

- *x(s)* の部分が *i(n)x(n)* に相当するからこれらを合わせるとダウンサンプリングのz変換表現を得る

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}} \exp(-j2\pi k/M))$$

Perfect Reconstruction (4)

- Decimation to 1/M by z-transform

$$\sum_{n=-\infty}^{\infty} x(nM)z^{-n} = \sum_{s=-\infty}^{\infty} x(s)z^{\frac{1}{M}(-s)} = X(z^{\frac{1}{M}})$$

$$s \equiv nM$$

$$n = \frac{1}{M}s$$

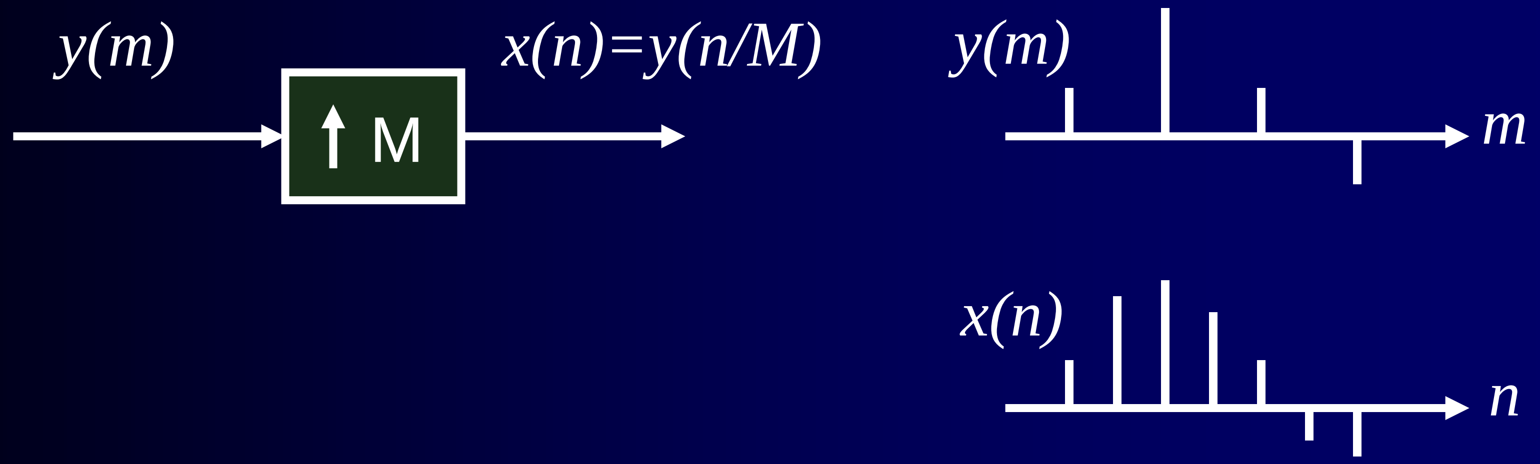
- *x(s)* corresponds to *i(n)x(n)*, so that we have z-transform of down sampling

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}}\exp(-j2\pi k/M))$$

完全再構成 (5)

- アップサンプルのz変換表現

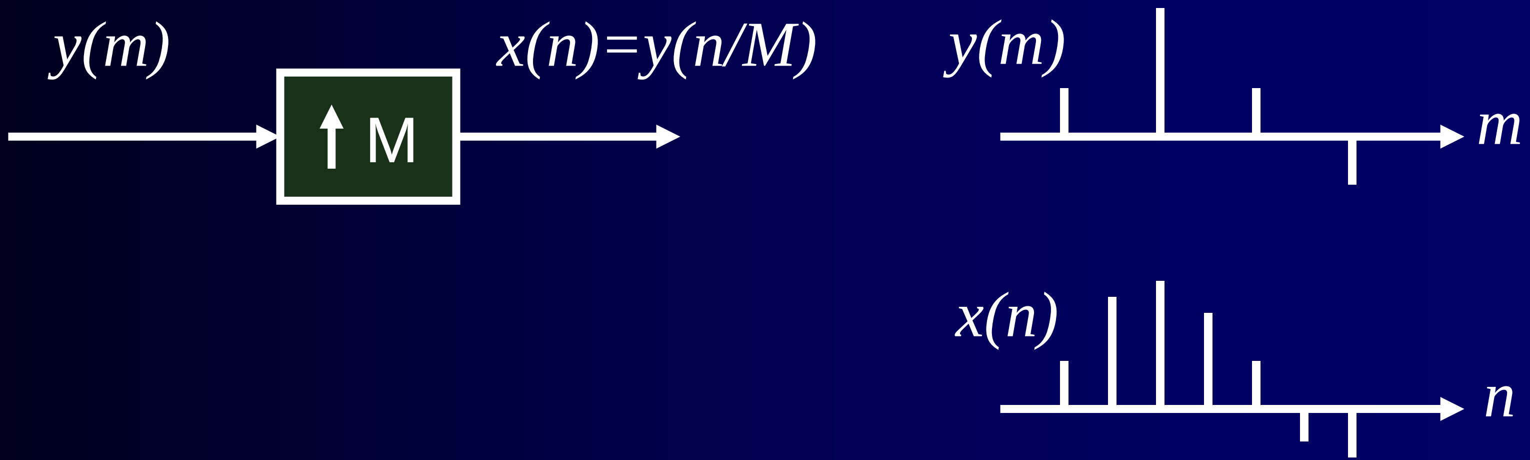
$$X(z) = Y(z^M)$$



Perfect Reconstruction (5)

- Z-transform representation for up sampling

$$X(z)=Y(z^{M})$$



完全再構成 (6)

- なぜなら，M倍にサンプルすると，z変換では以下のように表される

$$\sum_{n=-\infty}^{\infty} x(n/M)z^{-n} = \sum_{s=-\infty}^{\infty} x(s)z^{-Ms} = X(z^M)$$

$$s \equiv n/M$$

$$n = Ms$$

Perfect Reconstruction (6)

- Since M times up sampling is represented by z-transform as follows

$$\sum_{n=-\infty}^{\infty} x(n/M)z^{-n} = \sum_{s=-\infty}^{\infty} x(s)z^{-Ms} = X(z^M)$$

$$s \equiv n/M$$

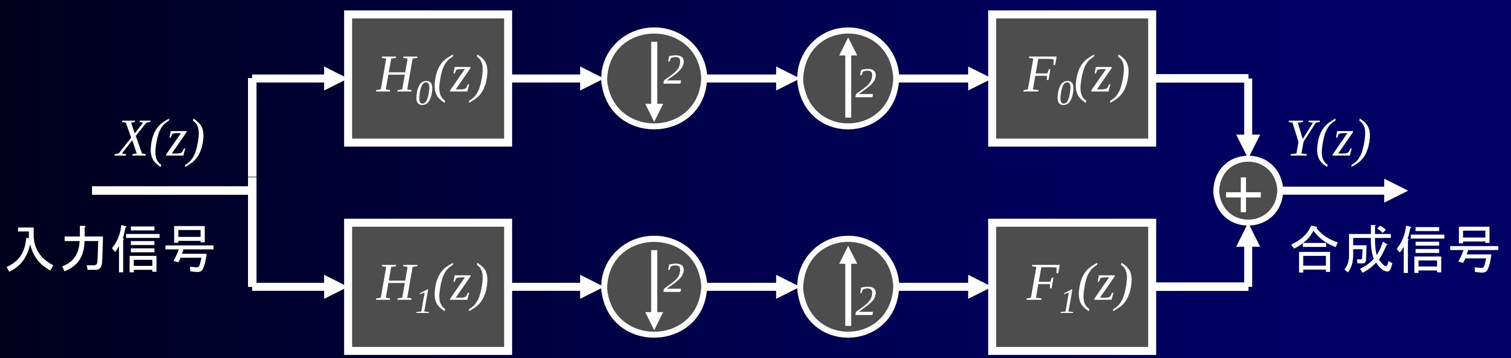
$$n = Ms$$

完全再構成 (7)

- フィルタリング，ダウンサンプリング，アップサンプリング，フィルタリングを縦続接続した信号をz変換で表現する

$$Y(z) = F_0(z)\frac{1}{2}\sum_{k=0}^{1} H_0(z e^{-j2\pi k/2})X(z e^{-j2\pi k/2})$$

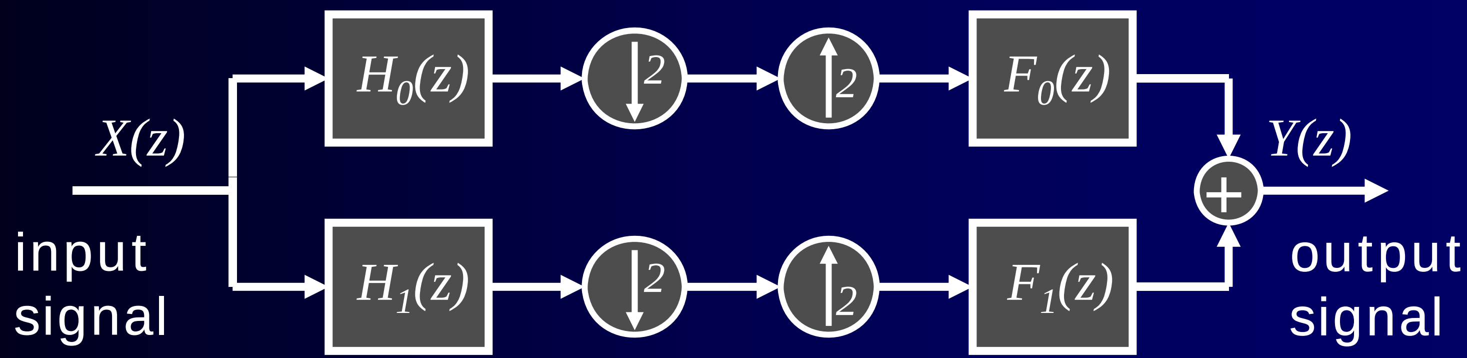
$$+ F_1(z)\frac{1}{2}\sum_{k=0}^{1} H_1(z e^{-j2\pi k/2})X(z e^{-j2\pi k/2})$$



Perfect Reconstruction (7)

- Cascade connection of filtering, down sampling, up sampling and filtering is represented by z-transform

$$Y(z) = F_0(z)\frac{1}{2}\sum_{k=0}^{1} H_0(ze^{-j2\pi k/2})X(ze^{-j2\pi k/2})$$
$$+F_1(z)\frac{1}{2}\sum_{k=0}^{1} H_1(ze^{-j2\pi k/2})X(ze^{-j2\pi k/2})$$



完全再構成 (8)

$$
\begin{aligned}
Y(z) &= F_0(z)\frac{1}{2}\sum_{k=0}^{1} H_0(ze^{-j2\pi k/2})X(ze^{-j2\pi k/2}) \\
&\quad + F_1(z)\frac{1}{2}\sum_{k=0}^{1} H_1(ze^{-j2\pi k/2})X(ze^{-j2\pi k/2}) \\
&= \frac{1}{2}F_0(z)(H_0(z)X(z)+H_0(-z)X(-z)) \\
&\quad + \frac{1}{2}F_1(z)(H_1(z)X(z)+H_1(-z)X(-z)) \\
&= \frac{1}{2}(F_0(z)H_0(z)+F_1(z)H_1(z))X(z) \\
&\quad + \frac{1}{2}(F_0(z)H_0(-z)+F_1(z)H_1(-z))X(-z)
\end{aligned}
$$

Perfect Reconstruction (8)

$$Y(z) = F_0(z)\frac{1}{2}\sum_{k=0}^{1} H_0(ze^{-j2\pi k/2})X(ze^{-j2\pi k/2})$$

$$+F_1(z)\frac{1}{2}\sum_{k=0}^{1} H_1(ze^{-j2\pi k/2})X(ze^{-j2\pi k/2})$$

$$=\frac{1}{2}F_0(z)(H_0(z)X(z)+H_0(-z)X(-z))$$

$$+\frac{1}{2}F_1(z)(H_1(z)X(z)+H_1(-z)X(-z))$$

$$=\frac{1}{2}(F_0(z)H_0(z)+F_1(z)H_1(z))X(z)$$

$$+\frac{1}{2}(F_0(z)H_0(-z)+F_1(z)H_1(-z))X(-z)$$

完全再構成 (9)

- 入出力が一致 ($Y(z)=X(z)$) するためには

$$Y(z)=\frac{1}{2}(F_0(z)H_0(z)+F_1(z)H_1(z))X(z)$$
$$+\frac{1}{2}(F_0(z)H_0(-z)+F_1(z)H_1(-z))X(-z)$$

において，1項目が適当な遅延だけを持ち，2項目が0であることが必要であるから

$$H_0(-z)F_0(z)+H_1(-z)F_1(z)=0$$
$$H_0(z)F_0(z)+H_1(z)F_1(z)=2z^{-L}$$

Perfect Reconstruction (9)

- For input and output matching, ($Y(z)=X(z)$)

$$Y(z)=\frac{1}{2}(F_0(z)H_0(z)+F_1(z)H_1(z))X(z)$$
$$+\frac{1}{2}(F_0(z)H_0(-z)+F_1(z)H_1(-z))X(-z)$$

We need arbitrary delay for the 1st term and 0 for the 2nd term

$$H_0(-z)F_0(z)+H_1(-z)F_1(z)=0$$
$$H_0(z)F_0(z)+H_1(z)F_1(z)=2z^{-L}$$

2バンドフィルタ

- QMF (Quadrature Mirror Filter)
  - 完全再構成ではない直線位相フィルタ
  - 周波数分離特性に優れる

$$H_1(z) = H_0(-z)$$

$$F_0(z) = H_1(-z) \qquad F_1(z) = -H_0(-z)$$

- CQF (Conjugate Quadrature Filter)
  - 完全再構成の非直線位相フィルタ(偶数次係数0のHalfband Filter)

$$H_1(z) = z^{-(N-1)} H_0(-z^{-1})$$

$$F_0(z) = H_1(-z) \qquad F_1(z) = H_0(-z)$$

2-band filter

- QMF (Quadrature Mirror Filter)
  - Linear phase filter, but not Perfect Reconstruction
  - Better band splitting

$$H_1(z) = H_0(-z)$$

$$F_0(z) = H_1(-z) \qquad F_1(z) = -H_0(-z)$$

- CQF (Conjugate Quadrature Filter)
  - Non linear phase, Perfect Reconstruction (even order coefficients are 0 in Halfband Filter)

$$H_1(z) = z^{-(N-1)} H_0(-z^{-1})$$

$$F_0(z) = H_1(-z) \qquad F_1(z) = H_0(-z)$$

2バンドフィルタ (2)

- SSKF (Symmetric Short Kernel Filter)
  - 直線位相の非直交フィルタ

$$F_0(z) = H_1(-z)$$
$$F_1(z) = -H_0(-z)$$

- SSKF(5,3)フィルタの係数の例

	n	係数値
$H_0(z)$	0,4	-0.125
	1,3	0.25
	2	0.75
$H_1(z)$	0,2	0.5
	1	-1.0

2-band filter (2)

- SSKF (Symmetric Short Kernel Filter)
  - Linear phase filter, not orthogonal

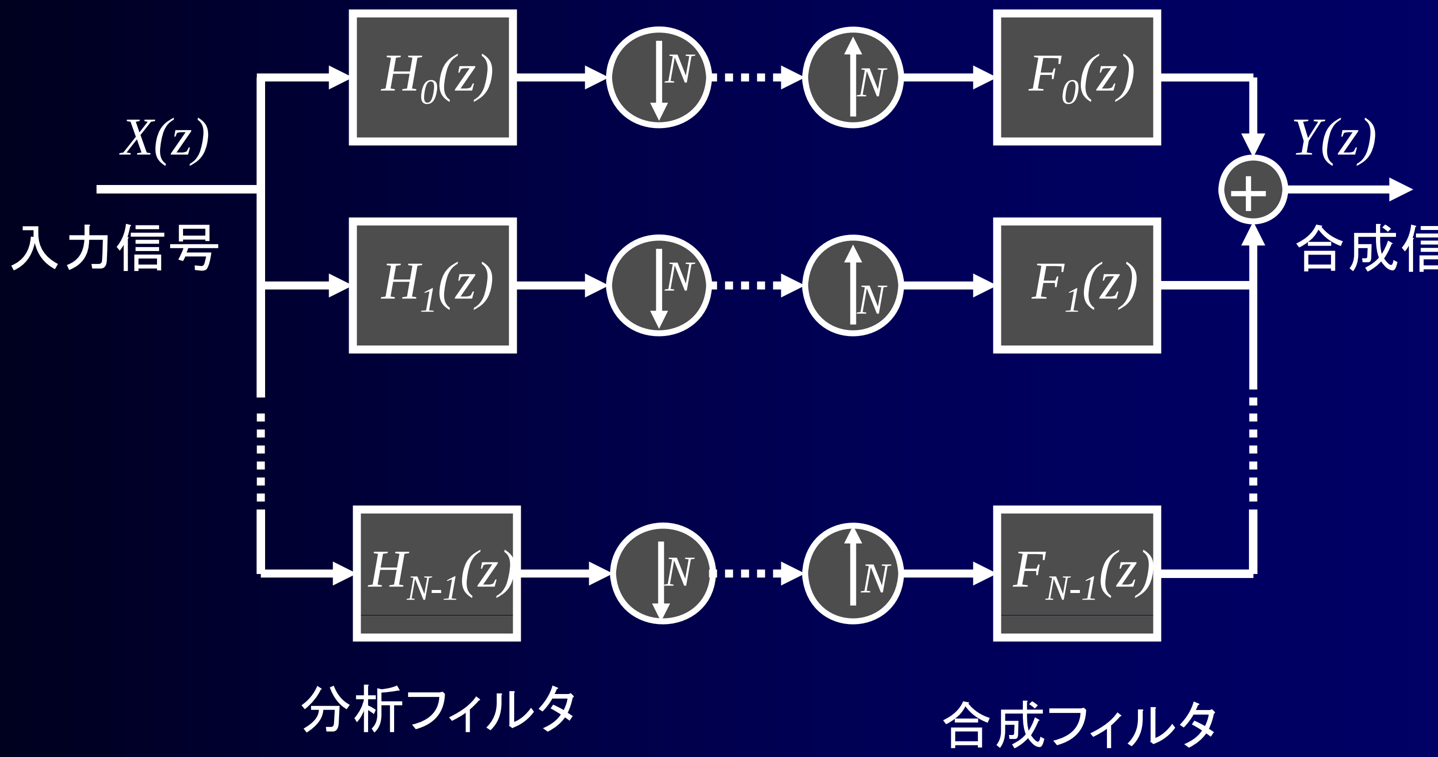
$$F_0(z) = H_1(-z)$$
$$F_1(z) = -H_0(-z)$$

- Example of SSKF(5,3) filter

	n	Coeff.
$H_0(z)$	0,4	-0.125
	1,3	0.25
	2	0.75
$H_1(z)$	0,2	0.5
	1	-1.0

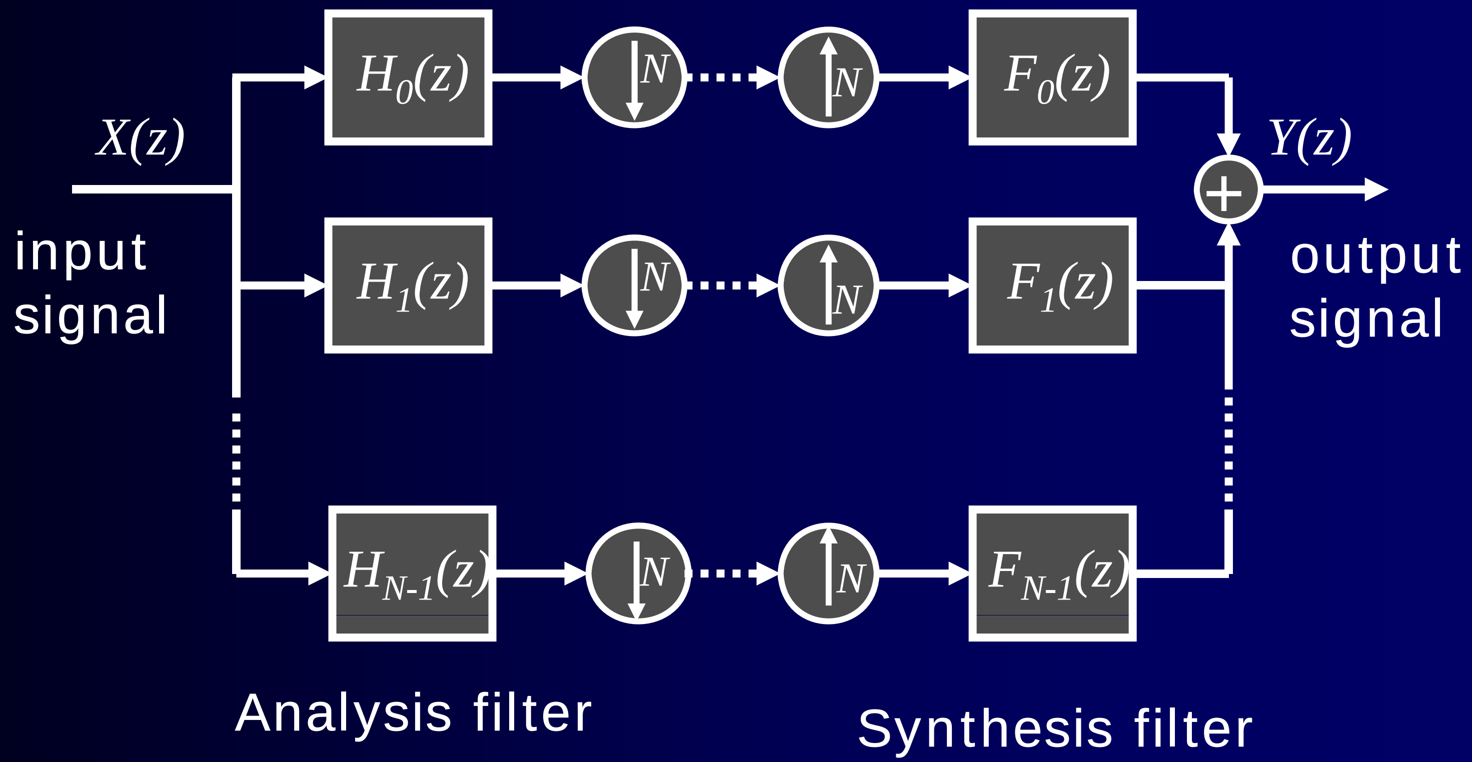
Nバンドフィルタ

- Nバンドフィルタバンクの構成



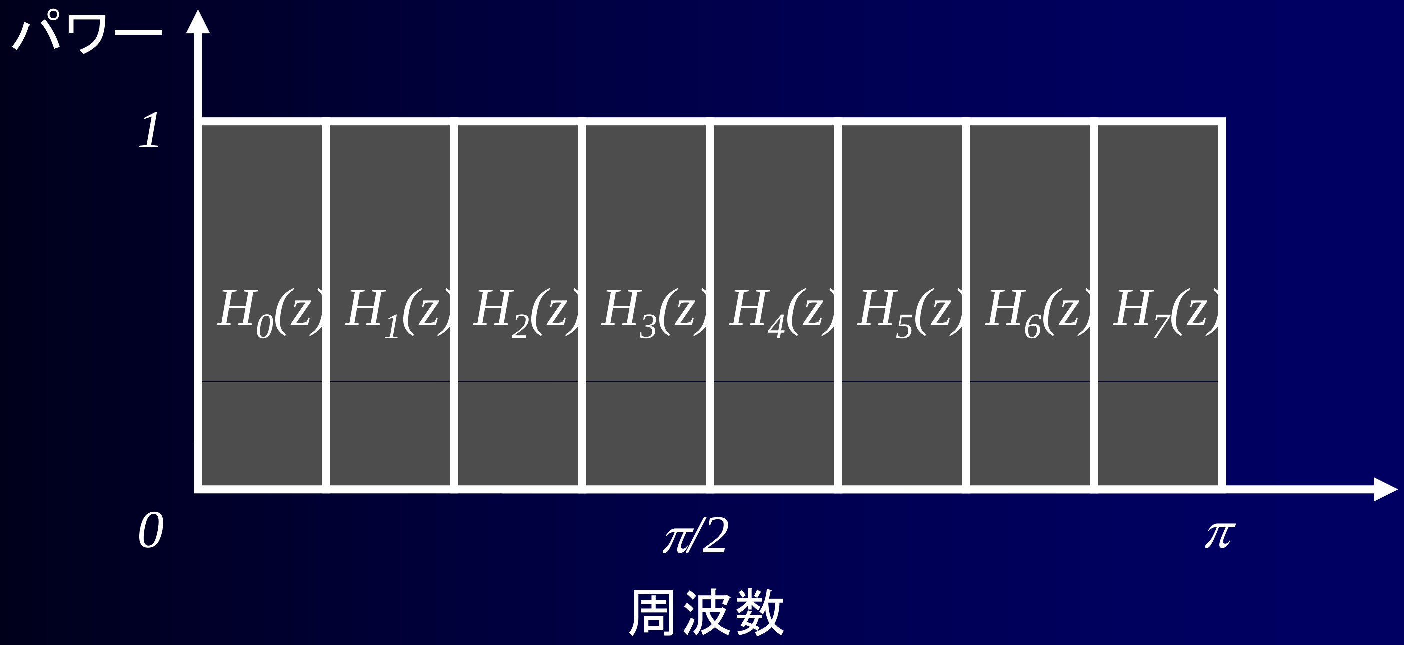
N-band filter

- Structure of N-band filterbank



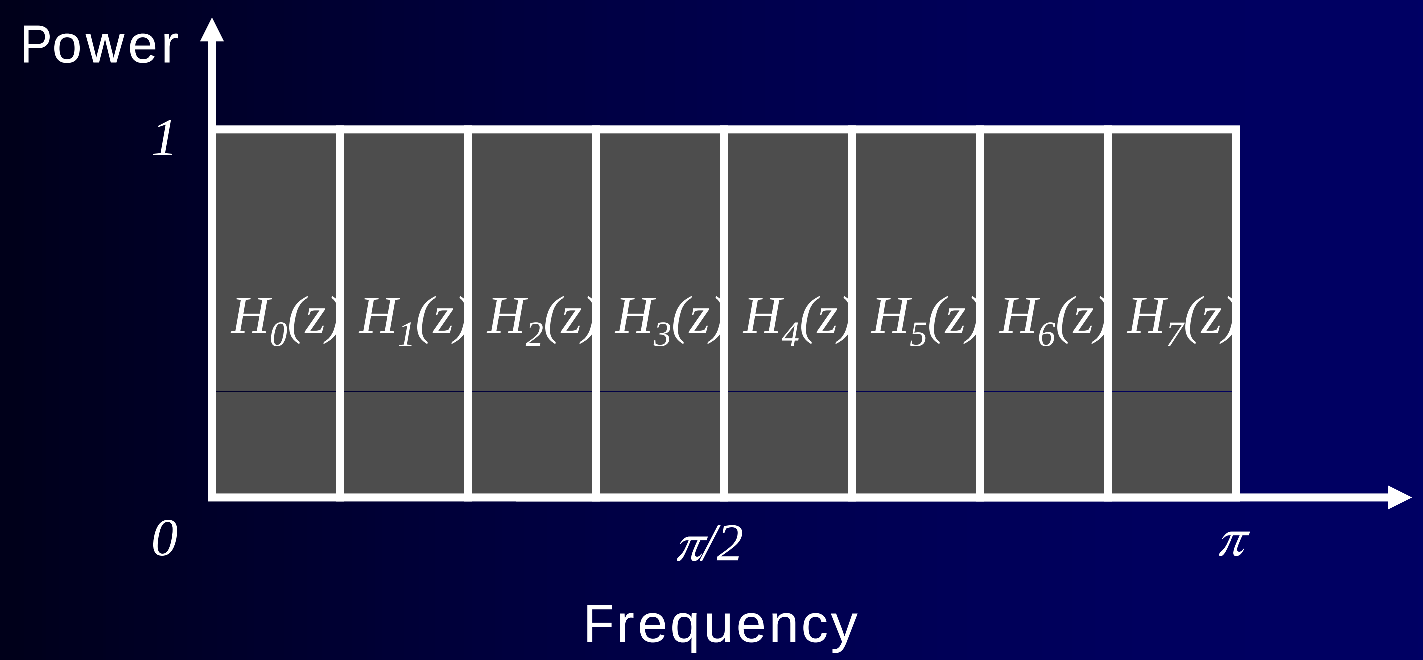
Nバンドフィルタ (2)

- 理想フィルタバンクの周波数特性 (8band)



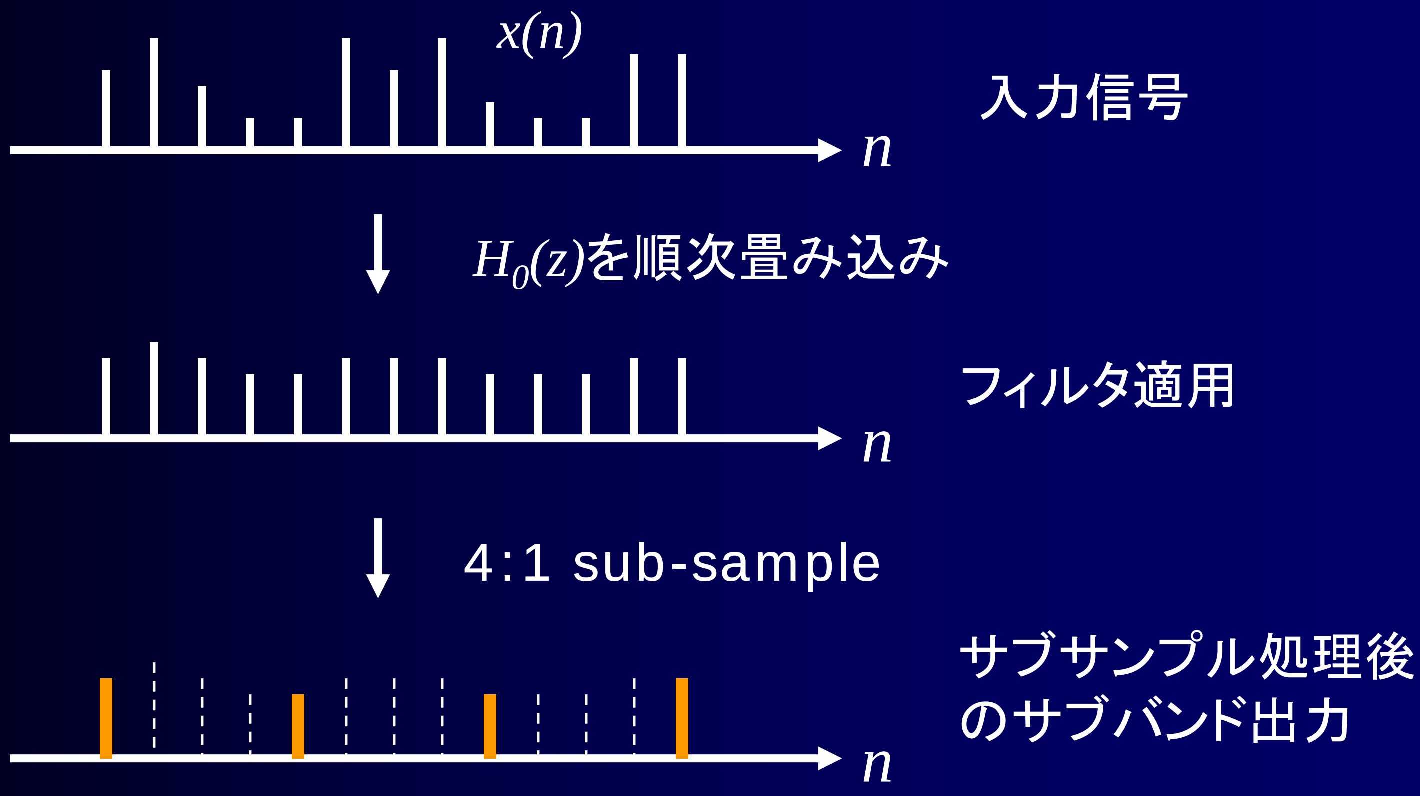
N-band filter (2)

- Ideal frequency response of filterbank (8-band)



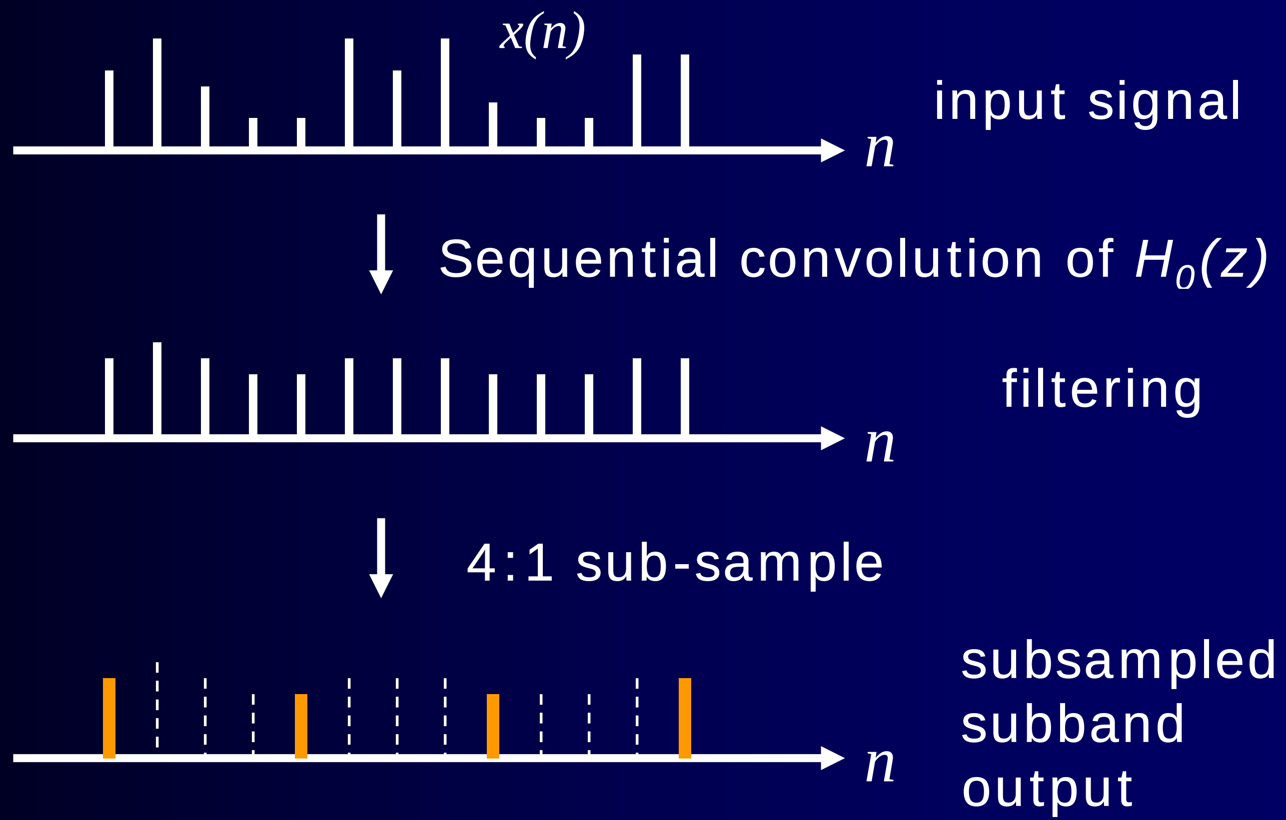
Nバンドフィルタ (3)

- 4バンドの場合のサブバンド出力



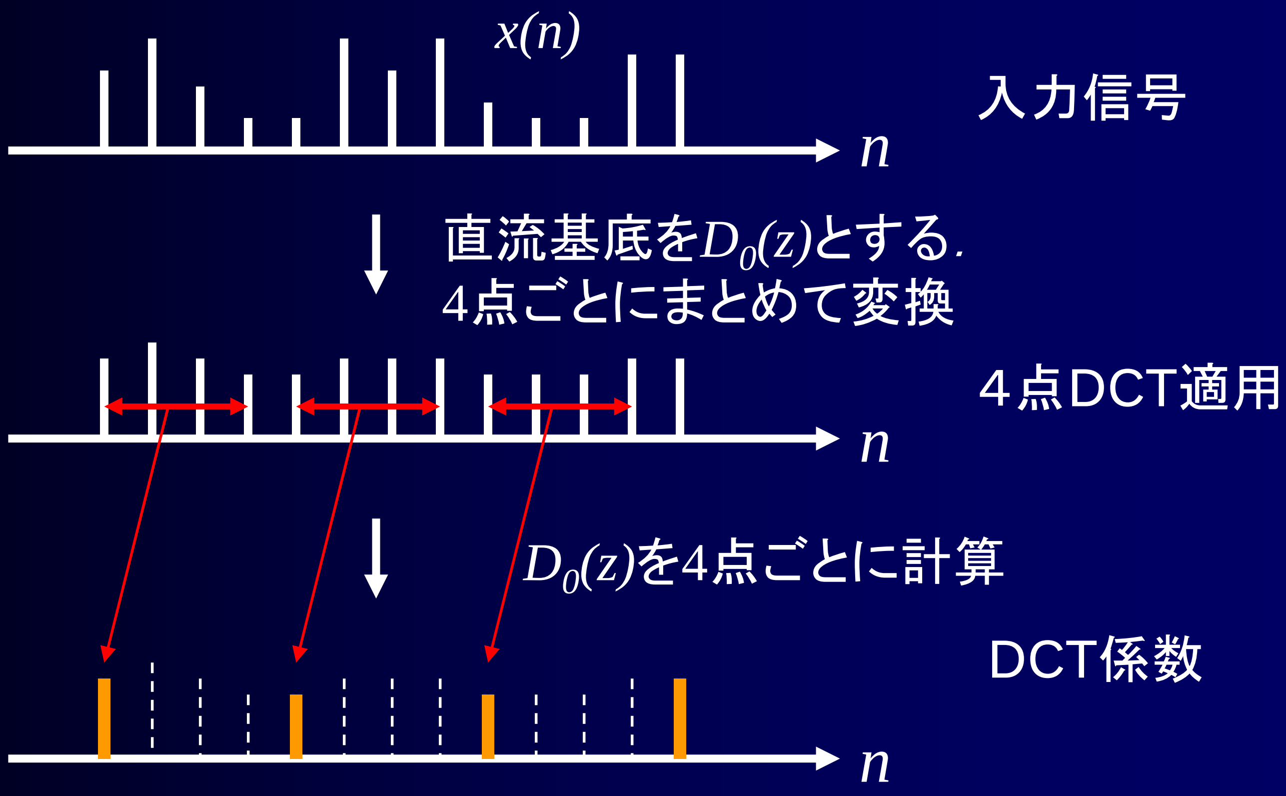
N-band filter (3)

- Subband output at 4-band



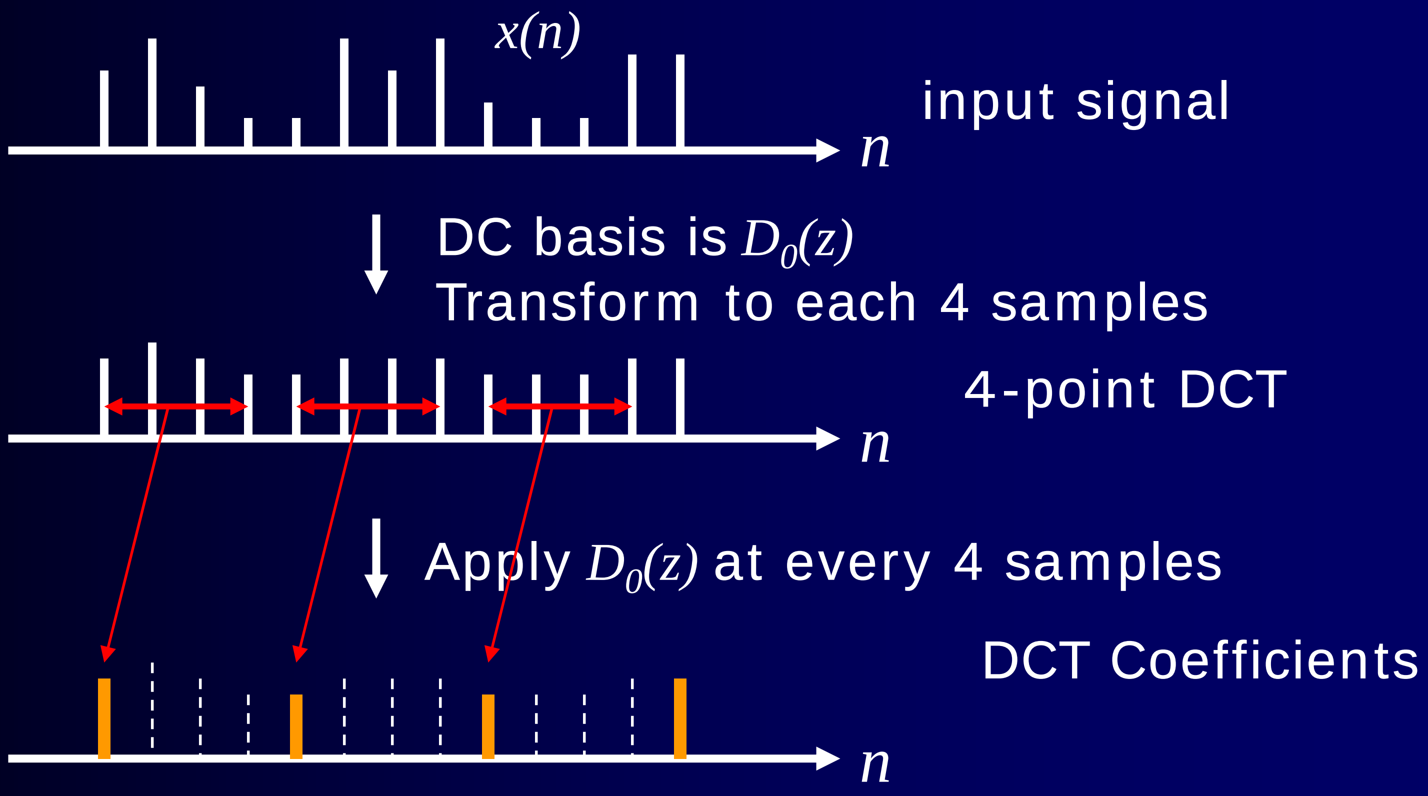
Nバンドフィルタ (4)

- 4点DCTの場合の係数



N-band filter (4)

- Coefficient of 4-point DCT

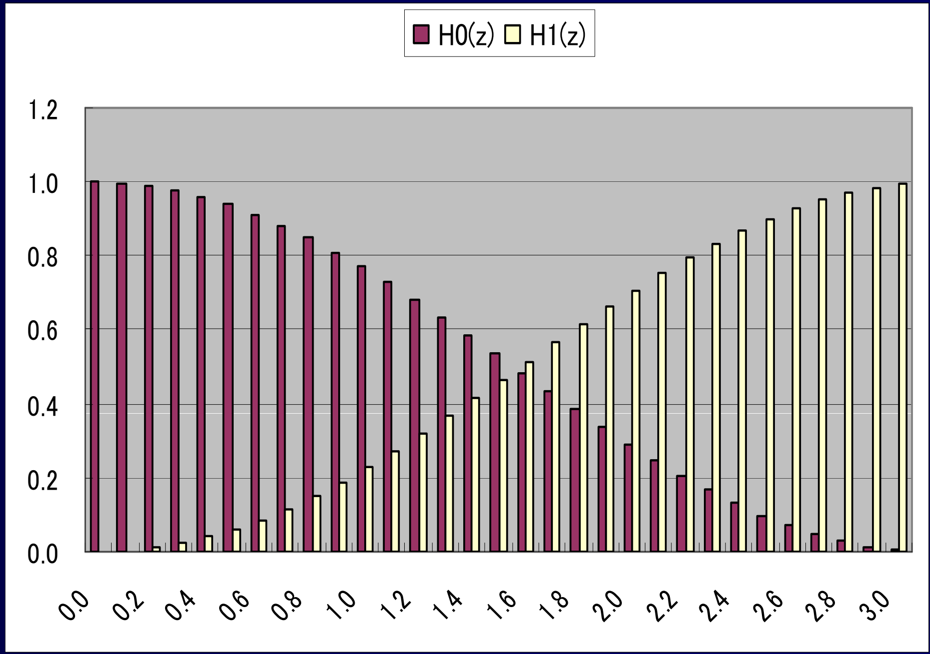


周波数特性の計算例

- Excel による周波数特性の計算

$$H_0(z) = \frac{1}{4}z^{-1} + \frac{1}{2} + \frac{1}{4}z^{1}$$

$$H_1(z) = -\frac{1}{4}z^{-1} + \frac{1}{2} + -\frac{1}{4}z^{1}$$



Example of Frequency Response Calculation

- Calculation of Frequency Response using Excel

$$H_0(z) = \frac{1}{4}z^{-1} + \frac{1}{2} + \frac{1}{4}z^{1}$$

$$H_1(z) = -\frac{1}{4}z^{-1} + \frac{1}{2} + -\frac{1}{4}z^{1}$$