

信号理論

- No.6 直交変換 -

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信号理論 / Signal Theory

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Signal Theory

- No.6 Orthogonal Transform -

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直交変換

- 直交変換: 線形変換の際の変換行列が直交行列
- 1次元線形変換

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- 行列表示

$$\mathbf{X} = \mathbf{A}\mathbf{x}$$

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Orthogonal Transform

- Orthogonal Transform: Transform matrix is orthogonal
- 1-dimensional linear transform

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- Matrix representation

$$\mathbf{X} = \mathbf{A}\mathbf{x}$$

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直交変換(2)

- 1次元N点線形変換

$$\mathbf{X} = \mathbf{A}\mathbf{x}$$

X: 変換係数ベクトル(Nx1)
A: 変換行列(NxN)
x: 入力ベクトル(Nx1)

- 逆行列の存在が必要

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{X}$$

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Orthogonal Transform(2)

- 1-dimensional N-point linear transform

$$\mathbf{X} = \mathbf{A}\mathbf{x}$$

X: Transform coefficient vector (Nx1)
A: Transform matrix (NxN)
x: input vector (Nx1)

- Presence of Inverse matrix is necessary

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{X}$$

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直交変換(3)

- 変換行列が直交行列
 - 任意の行ベクトルと行ベクトルが直交
 - 転置行列と逆行列が等しい

$$\mathbf{A}^t = \mathbf{A}^{-1}$$

Orthogonal Matrix(3)

- Transform matrix is orthogonal
 - Arbitrary row vector and other row vectors are orthogonal (column vectors have also the same property)
 - Transpose matrix equals to Inverse matrix

$$\mathbf{A}^t = \mathbf{A}^{-1}$$

復習

- 行列の転置: t (transpose)

$$\begin{bmatrix} m_0 \\ m_1 \end{bmatrix}^t = \begin{bmatrix} m_0 & m_1 \end{bmatrix}$$

$$\begin{bmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,n-1} \\ m_{1,0} & m_{1,1} & \cdots & m_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0} & m_{n-1,1} & \cdots & m_{n-1,n-1} \end{bmatrix}^t = \begin{bmatrix} m_{0,0} & m_{1,0} & \cdots & m_{n-1,0} \\ m_{0,1} & m_{1,1} & \cdots & m_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{0,n-1} & m_{1,n-1} & \cdots & m_{n-1,n-1} \end{bmatrix}$$

Review

- Transpose of matrix: t (transpose)

$$\begin{bmatrix} m_0 \\ m_1 \end{bmatrix}^t = \begin{bmatrix} m_0 & m_1 \end{bmatrix}$$

$$\begin{bmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,n-1} \\ m_{1,0} & m_{1,1} & \cdots & m_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0} & m_{n-1,1} & \cdots & m_{n-1,n-1} \end{bmatrix}^t = \begin{bmatrix} m_{0,0} & m_{1,0} & \cdots & m_{n-1,0} \\ m_{0,1} & m_{1,1} & \cdots & m_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{0,n-1} & m_{1,n-1} & \cdots & m_{n-1,n-1} \end{bmatrix}$$

直交変換(4)

- 2次元NxN次元直交変換

$$\mathbf{X}_{N \times N} = \mathbf{A} \mathbf{x}_{N \times N} \mathbf{A}^t$$

なぜなら

$$\begin{aligned} \mathbf{X}_{N \times N} &= \mathbf{A} \mathbf{x}_{N \times N} \mathbf{A}^t \\ &= [\mathbf{A} \mathbf{x}_0 \quad \mathbf{A} \mathbf{x}_1 \quad \cdots \quad \mathbf{A} \mathbf{x}_{N-1}] \mathbf{A}^t \\ &= \begin{bmatrix} \mathbf{A} \begin{bmatrix} [\mathbf{A} \mathbf{x}_0]^t \\ [\mathbf{A} \mathbf{x}_1]^t \\ \vdots \\ [\mathbf{A} \mathbf{x}_{N-1}]^t \end{bmatrix} \end{bmatrix}^t \end{aligned}$$

$\mathbf{X}$: 変換係数行列 (NxN)
 $\mathbf{A}$: 変換行列 (NxN)
 $\mathbf{x}$: 入力行列 (NxN)

Orthogonal Transform(4)

- 2-dimensional NxN point orthogonal transform

$$\mathbf{X}_{N \times N} = \mathbf{A} \mathbf{x}_{N \times N} \mathbf{A}^t$$

$\mathbf{X}$: Coefficient matrix (NxN)
 $\mathbf{A}$: Transform matrix (NxN)
 $\mathbf{x}$: Input matrix (NxN)

Since

$$\begin{aligned} \mathbf{X}_{N \times N} &= \mathbf{A} \mathbf{x}_{N \times N} \mathbf{A}^t \\ &= [\mathbf{A} \mathbf{x}_0 \quad \mathbf{A} \mathbf{x}_1 \quad \cdots \quad \mathbf{A} \mathbf{x}_{N-1}] \mathbf{A}^t \\ &= \begin{bmatrix} \mathbf{A} \begin{bmatrix} [\mathbf{A} \mathbf{x}_0]^t \\ [\mathbf{A} \mathbf{x}_1]^t \\ \vdots \\ [\mathbf{A} \mathbf{x}_{N-1}]^t \end{bmatrix} \end{bmatrix}^t \end{aligned}$$

直交変換(5)

- 離散フーリエ変換
- アダマール変換
- KL変換
- 離散サイン変換
- 離散コサイン変換

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Orthogonal Transform(5)

- Discrete Fourier Transform
- Hadamard Transform
- KL Transform
- Discrete Sine Transform
- Discrete Cosine Transform

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離散フーリエ変換

- N点離散フーリエ変換と逆変換

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk / N)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp(j2\pi nk / N)$$

離散的な角周波数

$$\omega_k = \frac{2\pi k}{N} \quad (k = 0, 1, \dots, N-1)$$

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Discrete Fourier Transform

- N-point Discrete Fourier Transform and Inverse transform

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk / N)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp(j2\pi nk / N)$$

Discrete frequency

$$\omega_k = \frac{2\pi k}{N} \quad (k = 0, 1, \dots, N-1)$$

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行列表現

- 1次元N点離散フーリエ変換

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & \cdots & e^{-j2\pi(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi(n-1)/N} & \cdots & e^{-j2\pi(N-1)(N-1)/N} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- 行列表現

$$\mathbf{X} = \mathbf{F}\mathbf{x}$$

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Matrix representation

- 1-dimensional N-point DFT

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & \cdots & e^{-j2\pi(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi(n-1)/N} & \cdots & e^{-j2\pi(N-1)(N-1)/N} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- Matrix representation

$$\mathbf{X} = \mathbf{F}\mathbf{x}$$

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離散フーリエ変換の特徴

- 離散フーリエ変換係数
 - N 点の整数値から N 点の複素数への変換
 - 変換係数は N 個の実数と虚数
 - 符号化すべきデータ個数は2倍に増加し、データ圧縮には不適
 - 直流成分は係数1個で表現される

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Characteristics of DFT

- DFT coefficients
 - From N -point integer to N -point complex value
 - Coefficients are N real numbers and N imaginary numbers
 - the number of data for coding application increased to 2 times, thus not suitable for data compression
 - DC component can be represented by one coefficient

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アダマール変換

- 変換マトリクスの要素が $[+1, -1]$ からなる最も簡単な直交変換
- ハードウェア化が容易
- $2n$ 次の変換マトリクスは、 $2 \times 2 (n=1)$ の基本マトリクスから拡張

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \mathbf{H}_2 \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \quad \mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- 拡張規則

$$\mathbf{H}_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & -\mathbf{H}_n \end{bmatrix}$$

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Hadamard Transform

- Element of transform matrix are $[+1, -1]$
- Simple orthogonal matrix, easy hardware realization
- $2n$ -th transform matrix can be derived from $2 \times 2 (n=1)$ -th basic matrix

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \mathbf{H}_2 \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \quad \mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Expansion rule

$$\mathbf{H}_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & -\mathbf{H}_n \end{bmatrix}$$

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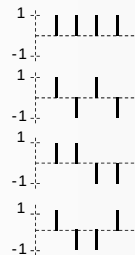
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アダマール変換(2)

- H_4, H_8

$$\mathbf{H}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} h(0)^t \\ h(1)^t \\ \vdots \\ h(7)^t \end{bmatrix}$$



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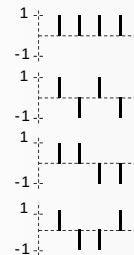
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Hadamard Transform(2)

- H_4, H_8

$$\mathbf{H}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} h(0)^t \\ h(1)^t \\ \vdots \\ h(7)^t \end{bmatrix}$$



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アダマール変換(3)

- H_8 のベクトルの要素

$$\begin{aligned} \mathbf{h}(0)^t &= [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1] \\ \mathbf{h}(1)^t &= [1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1] \\ \mathbf{h}(2)^t &= [1 \ 1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1] \\ \mathbf{h}(3)^t &= [1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1] \\ \mathbf{h}(4)^t &= [1 \ 1 \ 1 \ 1 \ -1 \ -1 \ -1 \ -1] \\ \mathbf{h}(5)^t &= [1 \ -1 \ 1 \ -1 \ -1 \ 1 \ -1 \ 1] \\ \mathbf{h}(6)^t &= [1 \ 1 \ -1 \ -1 \ -1 \ -1 \ 1 \ 1] \\ \mathbf{h}(7)^t &= [1 \ -1 \ -1 \ 1 \ -1 \ 1 \ 1 \ -1] \end{aligned}$$

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Hadamard Transform(3)

- vector element of H_8

$$\begin{aligned} \mathbf{h}(0)^t &= [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1] \\ \mathbf{h}(1)^t &= [1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1] \\ \mathbf{h}(2)^t &= [1 \ 1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1] \\ \mathbf{h}(3)^t &= [1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1] \\ \mathbf{h}(4)^t &= [1 \ 1 \ 1 \ 1 \ -1 \ -1 \ -1 \ -1] \\ \mathbf{h}(5)^t &= [1 \ -1 \ 1 \ -1 \ -1 \ 1 \ -1 \ 1] \\ \mathbf{h}(6)^t &= [1 \ 1 \ -1 \ -1 \ -1 \ -1 \ 1 \ 1] \\ \mathbf{h}(7)^t &= [1 \ -1 \ -1 \ 1 \ -1 \ 1 \ 1 \ -1] \end{aligned}$$

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ウォルシュ変換

- W_8 の変換マトリクスの要素は
アダマール変換の順序を変えたもの

$$\mathbf{W}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} \mathbf{h}(0)^t \\ \mathbf{h}(4)^t \\ \mathbf{h}(6)^t \\ \mathbf{h}(2)^t \\ \mathbf{h}(3)^t \\ \mathbf{h}(7)^t \\ \mathbf{h}(5)^t \\ \mathbf{h}(1)^t \end{bmatrix}$$

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Walsh Transform

- Elements of Walsh Transform
 W_8 are represented by different
order of Hadamard Transform's
elements

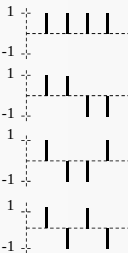
$$\mathbf{W}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} \mathbf{h}(0)^t \\ \mathbf{h}(4)^t \\ \mathbf{h}(6)^t \\ \mathbf{h}(2)^t \\ \mathbf{h}(3)^t \\ \mathbf{h}(7)^t \\ \mathbf{h}(5)^t \\ \mathbf{h}(1)^t \end{bmatrix}$$

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ウォルシュ変換の例

- W_4 による変換の例

$$\begin{aligned} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} x(0) + x(1) + x(2) + x(3) \\ x(0) + x(1) - x(2) - x(3) \\ x(0) - x(1) - x(2) + x(3) \\ x(0) - x(1) + x(2) - x(3) \end{bmatrix} \end{aligned}$$


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Example of Walsh Transform

- Example of Transform by W_4

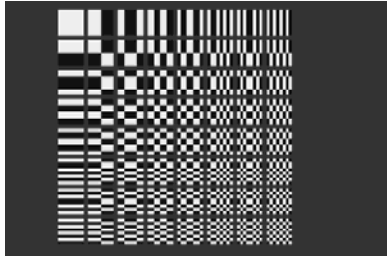
$$\begin{aligned} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} x(0) + x(1) + x(2) + x(3) \\ x(0) + x(1) - x(2) - x(3) \\ x(0) - x(1) - x(2) + x(3) \\ x(0) - x(1) + x(2) - x(3) \end{bmatrix} \end{aligned}$$


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基底画像

- 8x8ウォルシュ変換の基底画像

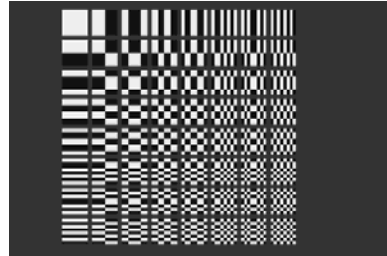


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Picture of Walsh Basis

- Picture of 8x8 Walsh Transform basis



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アダマール変換の特徴

- 構成が簡単
- 基底の形状が階段状であり、なだらかな波形の近似に向かない
- 信号の圧縮に用いると、階段状(ブロック状)の雑音を生じる
- 圧縮効率は低い

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Characteristics of Hadamard Transform

- Very simple structure
- Shape of basis is step like, thus difficult to approximate smoothly changing signal
- Cause step like (block) noise for signal compression
- Low compression efficiency

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KL(Karhunen-Loeve)変換

- 入力信号 $x(n)$ の自己相関行列 R_{xx} を対角化
- 対角化の際の固有ベクトルを基底ベクトルとする変換

$$\mathbf{R}_{xx} = E \left[\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix} \right]$$

$$= \begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r(1) & r(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r(1) \\ r(N-1) & \cdots & r(1) & r(0) \end{bmatrix}$$

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KL(Karhunen-Loeve) Transform

- Diagonalize auto-correlation matrix R_{xx} of input signal $x(n)$
- Basis vector of transform is eigenvector

$$\mathbf{R}_{xx} = E \left[\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix} \right]$$

$$= \begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r(1) & r(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r(1) \\ r(N-1) & \cdots & r(1) & r(0) \end{bmatrix}$$

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KL(Karhunen-Loeve)変換(2)

- 固有ベクトル t_n を要素とする行列 T による対角化

$$\mathbf{TR}_{xx}\mathbf{T}^t = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{N-1}^2 \end{bmatrix}$$

$$\mathbf{T}^t = \begin{bmatrix} t_0(0) & t_1(0) & \cdots & t_{N-1}(0) \\ t_0(1) & t_1(1) & \cdots & t_{N-1}(1) \\ \vdots & \vdots & \cdots & \vdots \\ t_0(N-1) & t_1(N-1) & \cdots & t_{N-1}(N-1) \end{bmatrix}$$

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KL(Karhunen-Loeve)Transform(2)

- Diagonalize by matrix T which elements are eigenvector t_n

$$\mathbf{TR}_{xx}\mathbf{T}^t = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{N-1}^2 \end{bmatrix}$$

$$\mathbf{T}^t = \begin{bmatrix} t_0(0) & t_1(0) & \cdots & t_{N-1}(0) \\ t_0(1) & t_1(1) & \cdots & t_{N-1}(1) \\ \vdots & \vdots & \cdots & \vdots \\ t_0(N-1) & t_1(N-1) & \cdots & t_{N-1}(N-1) \end{bmatrix}$$

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KL(Karhunen-Loeve)変換(3)

- KL変換係数 $k(n)$ ($n=0, \dots, N-1$) は入力信号を $x(n)$ ($n=0, \dots, N-1$) とすると次式で与えられる

$$\begin{bmatrix} k(0) \\ k(1) \\ \vdots \\ k(N-1) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} t_0(0) & t_0(1) & \cdots & t_0(N-1) \\ t_1(0) & t_1(1) & \cdots & t_1(N-1) \\ \cdots & \cdots & \cdots & \cdots \\ t_{N-1}(0) & t_{N-1}(1) & \cdots & t_{N-1}(N-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

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KL(Karhunen-Loeve)Transform(3)

- KL transform coefficient $k(n)$ ($n=0, \dots, N-1$) for input signal $x(n)$ ($n=0, \dots, N-1$) can be obtained by

$$\begin{bmatrix} k(0) \\ k(1) \\ \vdots \\ k(N-1) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} t_0(0) & t_0(1) & \cdots & t_0(N-1) \\ t_1(0) & t_1(1) & \cdots & t_1(N-1) \\ \cdots & \cdots & \cdots & \cdots \\ t_{N-1}(0) & t_{N-1}(1) & \cdots & t_{N-1}(N-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

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KL変換係数

- KL変換係数は互いに無相関

$$\mathbf{k} = \mathbf{T}\mathbf{x}$$

$$\begin{aligned} E[\mathbf{k}\mathbf{k}^t] &= E[\mathbf{T}\mathbf{x}(\mathbf{T}\mathbf{x})^t] \\ &= E[\mathbf{T}\mathbf{x}\mathbf{x}^t\mathbf{T}^t] \\ &= \mathbf{T}E[\mathbf{x}\mathbf{x}^t]\mathbf{T}^t \\ &= \mathbf{TR}_{xx}\mathbf{T}^t \\ &= \text{diag}[\sigma_0^2 \quad \sigma_1^2 \quad \cdots \quad \sigma_{N-1}^2] \end{aligned}$$

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KL Transform Coefficient

- KL Transform coefficients are not correlated each other

$$\mathbf{k} = \mathbf{T}\mathbf{x}$$

$$\begin{aligned} E[\mathbf{k}\mathbf{k}^t] &= E[\mathbf{T}\mathbf{x}(\mathbf{T}\mathbf{x})^t] \\ &= E[\mathbf{T}\mathbf{x}\mathbf{x}^t\mathbf{T}^t] \\ &= \mathbf{T}E[\mathbf{x}\mathbf{x}^t]\mathbf{T}^t \\ &= \mathbf{TR}_{xx}\mathbf{T}^t \\ &= \text{diag}[\sigma_0^2 \quad \sigma_1^2 \quad \cdots \quad \sigma_{N-1}^2] \end{aligned}$$

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KL変換係数(2)

- 無相関とは...

$$E[k(n)k(m)] \begin{cases} = 0 & (n \neq m) \\ \neq 0 & (n = m) \end{cases}$$

$$E[\mathbf{k}\mathbf{k}^T] = \begin{bmatrix} E[k(0)k(0)] & E[k(0)k(1)] & \cdots & E[k(0)k(N-1)] \\ E[k(1)k(0)] & E[k(1)k(1)] & \cdots & E[k(1)k(N-1)] \\ \vdots & \vdots & \ddots & \vdots \\ E[k(N-1)k(0)] & E[k(N-1)k(1)] & \cdots & E[k(N-1)k(N-1)] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{N-1}^2 \end{bmatrix}$$

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KL Transform Coefficient(2)

- What is "not correlated"?

$$E[k(n)k(m)] \begin{cases} = 0 & (n \neq m) \\ \neq 0 & (n = m) \end{cases}$$

$$E[\mathbf{k}\mathbf{k}^T] = \begin{bmatrix} E[k(0)k(0)] & E[k(0)k(1)] & \cdots & E[k(0)k(N-1)] \\ E[k(1)k(0)] & E[k(1)k(1)] & \cdots & E[k(1)k(N-1)] \\ \vdots & \vdots & \ddots & \vdots \\ E[k(N-1)k(0)] & E[k(N-1)k(1)] & \cdots & E[k(N-1)k(N-1)] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{N-1}^2 \end{bmatrix}$$

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復習

- 行列 A の対角化には固有ベクトル $\mathbf{t}$ が必要

$$\mathbf{A}\mathbf{t} = \lambda\mathbf{t}$$

- どうやって固有値/を求めるか? 固有方程式を計算する.

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{t} = \mathbf{0}$$

- 固有値は, 左辺の行列式を解いて得る

$$|\lambda\mathbf{I} - \mathbf{A}| = 0$$

- 対角化する行列は固有ベクトルを並べたもの

$$\mathbf{T} = [\mathbf{t}_0 \quad \mathbf{t}_1 \quad \cdots \quad \mathbf{t}_{N-1}]$$

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Review

- We need eigenvector $\mathbf{t}$ to diagonalize matrix A

$$\mathbf{A}\mathbf{t} = \lambda\mathbf{t}$$

- How to obtain eigenvalue λ ? Characteristic equation.

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{t} = \mathbf{0}$$

- Eigenvalue can be obtained by solving determinant of the left term

$$|\lambda\mathbf{I} - \mathbf{A}| = 0$$

- Matrix for diagonalization is a series of eigenvector

$$\mathbf{T} = [\mathbf{t}_0 \quad \mathbf{t}_1 \quad \cdots \quad \mathbf{t}_{N-1}]$$

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KL変換の特徴

- 低次のシーケンスに対するパワー集中は理論的に最高
- 入力信号の性質によって相関行列が異なる
- そのため、入力信号ごとにKL変換基底が異なる
- 受信側に基底ベクトルを送信する必要がある
- 理論限界を与えるものであり、実用的ではない

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Characteristics of KL Transform

- Theoretically best performance to compact energy to the low sequences
- Autocorrelation matrix differs depends on input signal
- Thus, KL transform basis differs depends on each input
- Needs to transmit basis vector to the receiver
- Gives theoretical bound, not practical

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離散サイン変換

- N点離散サイン変換(DST-II)と逆変換

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=1}^N x(n) \sin\left(\frac{(2n-1)k\pi}{2N}\right) \quad (k=1, \dots, N)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=1}^N C(k) X(k) \sin\left(\frac{(2n-1)k\pi}{2N}\right) \quad (n=1, \dots, N)$$

ここに

$$C(k) = \begin{cases} \frac{1}{\sqrt{2}} & (k=N) \\ 1 & (k \neq N) \end{cases}$$

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Discrete Sine Transform

- N-point Discrete Sine Transform(DST-II) and Inverse DST

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=1}^N x(n) \sin\left(\frac{(2n-1)k\pi}{2N}\right) \quad (k=1, \dots, N)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=1}^N C(k) X(k) \sin\left(\frac{(2n-1)k\pi}{2N}\right) \quad (n=1, \dots, N)$$

where

$$C(k) = \begin{cases} \frac{1}{\sqrt{2}} & (k=N) \\ 1 & (k \neq N) \end{cases}$$

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離散サイン変換(2)

- DST-I

$$\sin\left(\frac{nk\pi}{N}\right) \quad (k, n=1, 2, \dots, N-1)$$

- DST-II

$$C(k) \sin\left(\frac{(2n-1)k\pi}{2N}\right) \quad (k, n=1, 2, \dots, N)$$

- DST-III

$$C(n) \sin\left(\frac{(2k-1)n\pi}{2N}\right) \quad (k, n=1, 2, \dots, N)$$

- DST-IV

$$\sin\left(\frac{(2k+1)(2n+1)\pi}{4N}\right) \quad (k, n=0, 1, \dots, N-1)$$

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Discrete Sine Transform(2)

- DST-I

$$\sin\left(\frac{nk\pi}{N}\right) \quad (k, n=1, 2, \dots, N-1)$$

- DST-II

$$C(k) \sin\left(\frac{(2n-1)k\pi}{2N}\right) \quad (k, n=1, 2, \dots, N)$$

- DST-III

$$C(n) \sin\left(\frac{(2k-1)n\pi}{2N}\right) \quad (k, n=1, 2, \dots, N)$$

- DST-IV

$$\sin\left(\frac{(2k+1)(2n+1)\pi}{4N}\right) \quad (k, n=0, 1, \dots, N-1)$$

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離散サイン変換(3)

- 第1係数の計算

$$X(1) = \sqrt{\frac{2}{N}} C(1) \sum_{n=1}^N x(n) \sin\left(\frac{(2n-1)\pi}{2N}\right)$$

- 4点DSTの場合の第1番目の基底

$$\sqrt{\frac{2}{N}} C(1) \sin\left(\frac{(2n-1)\pi}{2N}\right) \quad (n=1, 2, 3, 4)$$

$$= \sqrt{\frac{1}{2}} \begin{bmatrix} \sin \frac{\pi}{8} & \sin \frac{3\pi}{8} & \sin \frac{5\pi}{8} & \sin \frac{7\pi}{8} \end{bmatrix}$$


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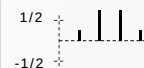
Discrete Sine Transform(3)

- Calculation of the first coefficient

$$X(1) = \sqrt{\frac{2}{N}} C(1) \sum_{n=1}^N x(n) \sin\left(\frac{(2n-1)\pi}{2N}\right)$$

- The first basis of 4-point DST

$$\sqrt{\frac{2}{N}} C(1) \sin\left(\frac{(2n-1)\pi}{2N}\right) \quad (n=1, 2, 3, 4)$$

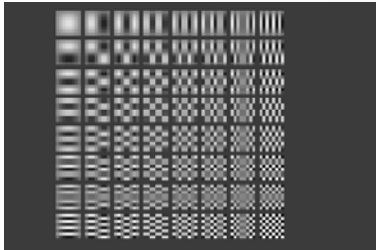
$$= \sqrt{\frac{1}{2}} \begin{bmatrix} \sin \frac{\pi}{8} & \sin \frac{3\pi}{8} & \sin \frac{5\pi}{8} & \sin \frac{7\pi}{8} \end{bmatrix}$$


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基底画像

- 8x8DSTの基底画像

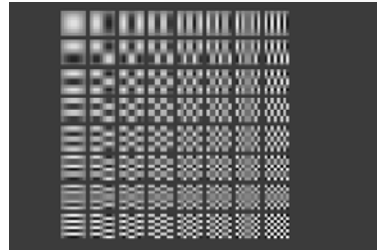


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Picture of DST Basis

- Picture of 8x8 DST basis



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離散サイン変換の特徴

- 直流成分を1個の変換係数で表現できない欠点がある
- 基底ベクトルの値は、原理的に区間の端点で0
- 隣接区間とのデータの連続性は良い

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Characteristics of DST

- Cannot represent DC component by one coefficient
- Edge value of basis vector is zero
- Smooth connection of data with the neighbor section

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離散コサイン変換 (DCT)

- N点DCT(タイプII) ... 信号 $x(n)$, 係数 $X(k)$

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} C(k) X(k) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (n = 0, \dots, N-1)$$

ここに,

$$C(k) = \begin{cases} \frac{1}{\sqrt{2}} & (k = 0) \\ 1 & (k \neq 0) \end{cases}$$

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Discrete Cosine Transform (DCT)

- N point DCT(Type-II) ... Signal $x(n)$, Coefficient $X(k)$

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} C(k) X(k) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (n = 0, \dots, N-1)$$

where,

$$C(k) = \begin{cases} \frac{1}{\sqrt{2}} & (k = 0) \\ 1 & (k \neq 0) \end{cases}$$

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離散コサイン変換(2)

- DCT-I

$$C(k)C(n)\cos\left(\frac{nk\pi}{N}\right) \quad (k, n = 0, 1, \dots, N)$$

- DCT-II

$$C(k)\cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (k, n = 0, 1, \dots, N-1)$$

- DCT-III

$$C(n)\cos\left(\frac{(2k+1)n\pi}{2N}\right) \quad (k, n = 0, 1, \dots, N-1)$$

- DCT-IV

$$\cos\left(\frac{(2n+1)(2k+1)\pi}{4N}\right) \quad (k, n = 0, 1, \dots, N-1)$$

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Discrete Cosine Transform(2)

- DCT-I

$$C(k)C(n)\cos\left(\frac{nk\pi}{N}\right) \quad (k, n = 0, 1, \dots, N)$$

- DCT-II

$$C(k)\cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (k, n = 0, 1, \dots, N-1)$$

- DCT-III

$$C(n)\cos\left(\frac{(2k+1)n\pi}{2N}\right) \quad (k, n = 0, 1, \dots, N-1)$$

- DCT-IV

$$\cos\left(\frac{(2n+1)(2k+1)\pi}{4N}\right) \quad (k, n = 0, 1, \dots, N-1)$$

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2次元DCT

- NxN点2次元DCT ... 信号 $x(n,m)$, 係数 $X(u,v)$

$$X(u,v) = \frac{2}{N} C(u)C(v) \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(n,m) \cos\left(\frac{(2n+1)u\pi}{2N}\right) \cos\left(\frac{(2m+1)v\pi}{2N}\right)$$

$$x(n,m) = \frac{2}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} C(u)C(v) X(u,v) \cos\left(\frac{(2n+1)u\pi}{2N}\right) \cos\left(\frac{(2m+1)v\pi}{2N}\right)$$

ここに,

$$C(u), C(v) = \begin{cases} \frac{1}{\sqrt{2}} & (u=0, v=0) \\ 1 & (u \neq 0, v \neq 0) \end{cases}$$

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Two-Dimensional DCT

- NxN point 2-D DCT ... Signal $x(n,m)$, Coefficient $X(u,v)$

$$X(u,v) = \frac{2}{N} C(u)C(v) \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(n,m) \cos\left(\frac{(2n+1)u\pi}{2N}\right) \cos\left(\frac{(2m+1)v\pi}{2N}\right)$$

$$x(n,m) = \frac{2}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} C(u)C(v) X(u,v) \cos\left(\frac{(2n+1)u\pi}{2N}\right) \cos\left(\frac{(2m+1)v\pi}{2N}\right)$$

where,

$$C(u), C(v) = \begin{cases} \frac{1}{\sqrt{2}} & (u=0, v=0) \\ 1 & (u \neq 0, v \neq 0) \end{cases}$$

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2次元DCTの計算

- 変数 n, m に関して独立な1次元DCTの処理に分解
- 2回のN点DCT演算として計算できる

$$t(u,m) = \sqrt{\frac{2}{N}} C(u) \sum_{n=0}^{N-1} x(n,m) \cos\left(\frac{(2n+1)u\pi}{2N}\right)$$

$$X(u,v) = \sqrt{\frac{2}{N}} C(v) \sum_{m=0}^{N-1} t(u,m) \cos\left(\frac{(2m+1)v\pi}{2N}\right)$$

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Calculation of 2-D DCT

- Decompose to independent 1-D DCT with regard to variables n, m
- Calculated as 2 times N point DCT

$$t(u,m) = \sqrt{\frac{2}{N}} C(u) \sum_{n=0}^{N-1} x(n,m) \cos\left(\frac{(2n+1)u\pi}{2N}\right)$$

$$X(u,v) = \sqrt{\frac{2}{N}} C(v) \sum_{m=0}^{N-1} t(u,m) \cos\left(\frac{(2m+1)v\pi}{2N}\right)$$

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8x8DCT

- 8x8点2次元DCT ... 信号 $x(n,m)$, 係数 $X(u,v)$

$$X(u,v) = \frac{1}{4} C(u)C(v) \sum_{n=0}^7 \sum_{m=0}^7 x(n,m) \cos\left(\frac{(2n+1)u\pi}{16}\right) \cos\left(\frac{(2m+1)v\pi}{16}\right)$$

$$x(n,m) = \frac{1}{4} \sum_{u=0}^7 \sum_{v=0}^7 C(u)C(v) X(u,v) \cos\left(\frac{(2n+1)u\pi}{16}\right) \cos\left(\frac{(2m+1)v\pi}{16}\right)$$

ここに,

$$C(u), C(v) = \begin{cases} \frac{1}{\sqrt{2}} & (u=0, v=0) \\ 1 & (u \neq 0, v \neq 0) \end{cases}$$

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8x8DCT

- 8x8 point 2-D DCT ... Signal $x(n,m)$, Coefficient $X(u,v)$

$$X(u,v) = \frac{1}{4} C(u)C(v) \sum_{n=0}^7 \sum_{m=0}^7 x(n,m) \cos\left(\frac{(2n+1)u\pi}{16}\right) \cos\left(\frac{(2m+1)v\pi}{16}\right)$$

$$x(n,m) = \frac{1}{4} \sum_{u=0}^7 \sum_{v=0}^7 C(u)C(v) X(u,v) \cos\left(\frac{(2n+1)u\pi}{16}\right) \cos\left(\frac{(2m+1)v\pi}{16}\right)$$

where,

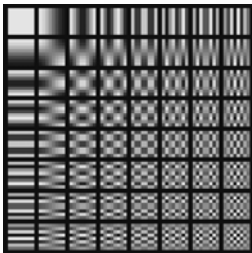
$$C(u), C(v) = \begin{cases} \frac{1}{\sqrt{2}} & (u=0, v=0) \\ 1 & (u \neq 0, v \neq 0) \end{cases}$$

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基底画像

- 8x8DCTの基底画像

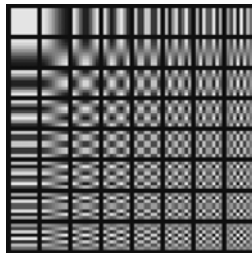


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Basis Image

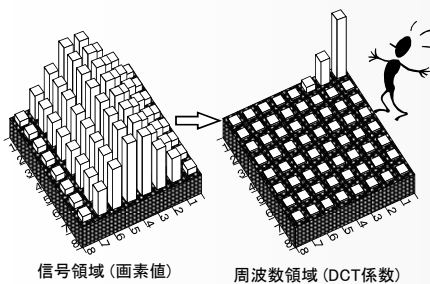
- Basis Image of 8x8DCT



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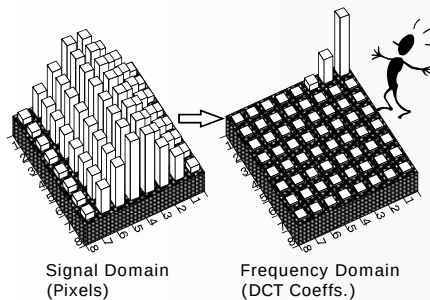
DCTの特徴



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Characteristics of DCT



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DCTの種類

■ DCT-I

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=0}^N C(n) x(n) \cos\left(\frac{nk\pi}{N}\right) \quad (k = 0, \dots, N)$$

$$x(n) = \sqrt{\frac{2}{N}} C(n) \sum_{k=0}^N C(k) X(k) \cos\left(\frac{nk\pi}{N}\right) \quad (n = 0, \dots, N)$$

■ DCT-II

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} C(k) X(k) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (n = 0, \dots, N-1)$$

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Type of DCT

■ DCT-I

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=0}^N C(n) x(n) \cos\left(\frac{nk\pi}{N}\right) \quad (k = 0, \dots, N)$$

$$x(n) = \sqrt{\frac{2}{N}} C(n) \sum_{k=0}^N C(k) X(k) \cos\left(\frac{nk\pi}{N}\right) \quad (n = 0, \dots, N)$$

■ DCT-II

$$X(k) = \sqrt{\frac{2}{N}} C(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} C(k) X(k) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \quad (n = 0, \dots, N-1)$$

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DCTの種類 (2)

■ DCT-III

$$X(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} C(n) x(n) \cos\left(\frac{(2k+1)n\pi}{2N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} C(n) \sum_{k=0}^{N-1} X(k) \cos\left(\frac{(2k+1)n\pi}{2N}\right) \quad (n = 0, \dots, N-1)$$

■ DCT-IV

$$X(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)(2k+1)\pi}{4N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} X(k) \cos\left(\frac{(2n+1)(2k+1)\pi}{4N}\right) \quad (n = 0, \dots, N-1)$$

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Type of DCT (2)

■ DCT-III

$$X(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} C(n) x(n) \cos\left(\frac{(2k+1)n\pi}{2N}\right) \quad (k = 0, \dots, N-1)$$

$$x(n) = \sqrt{\frac{2}{N}} C(n) \sum_{k=0}^{N-1} X(k) \cos\left(\frac{(2k+1)n\pi}{2N}\right) \quad (n = 0, \dots, N-1)$$

■ DCT-IV

$$X(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)(2k+1)\pi}{4N}\right) \quad (k = 0, \dots, N-1)$$

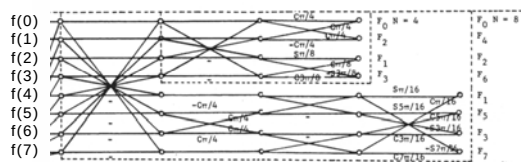
$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} X(k) \cos\left(\frac{(2n+1)(2k+1)\pi}{4N}\right) \quad (n = 0, \dots, N-1)$$

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DCTの高速算法

■ Chen-Smith-Fralick (1977)

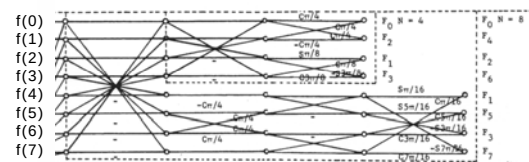


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Fast Algorithm for DCT

■ Chen-Smith-Fralick (1977)



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プログラミング

■ テーブル参照方式

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} t_{0,0} & t_{0,1} & \cdots & t_{0,N-1} \\ t_{1,0} & t_{1,1} & \cdots & t_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N-1,0} & t_{N-1,1} & \cdots & t_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}$$

■ プログラム

```
for(u=0; u<N; u++) {
    for(x=0; x<N; x++) F(u) +=t(u,x)*f(x);
}
```

Programming

■ Table Look Up

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} t_{0,0} & t_{0,1} & \cdots & t_{0,N-1} \\ t_{1,0} & t_{1,1} & \cdots & t_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N-1,0} & t_{N-1,1} & \cdots & t_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}$$

■ Program

```
for(u=0; u<N; u++) {
    for(x=0; x<N; x++) F(u) +=t(u,x)*f(x);
}
```

プログラミング(2)

- バタフライ演算による高速算法の場合
 - Chen-Smith-Fralickのフローグラフによる

■ プログラム

```
void dct_fast_1_8(short *data1, double *data2)
{
    short i;
    double a[8], b[8], c[8], d[8];
    double pai, c14, c18, s18, c38, s38, c116, s116, c316, s316
           c516, s516, c716, s716;
    pai=3.141592654;
    c14=cos(pai/4.0);
    c18=cos(pai/8.0);
    s18=sin(pai/8.0);
```

Programming(2)

- Fast Algorithm by Butterfly
 - Chen-Smith-Fralick's flowgraph

■ Program

```
void dct_fast_1_8(short *data1, double *data2)
{
    short i;
    double a[8], b[8], c[8], d[8];
    double pai, c14, c18, s18, c38, s38, c116, s116, c316, s316
           c516, s516, c716, s716;
    pai=3.141592654;
    c14=cos(pai/4.0);
    c18=cos(pai/8.0);
    s18=sin(pai/8.0);
```

プログラミング(3)

■ 続き

```
c38=cos(3.0*pai/8.0);
s38=sin(3.0*pai/8.0);
c116=cos(pai/16.0);
s116=sin(pai/16.0);
c316=cos(3.0*pai/16.0);
s316=sin(3.0*pai/16.0);
c516=cos(5.0*pai/16.0);
s516=sin(5.0*pai/16.0);
c716=cos(7.0*pai/16.0);
s716=sin(7.0*pai/16.0);
```

Programming(3)

■ Contd.

```
c38=cos(3.0*pai/8.0);
s38=sin(3.0*pai/8.0);
c116=cos(pai/16.0);
s116=sin(pai/16.0);
c316=cos(3.0*pai/16.0);
s316=sin(3.0*pai/16.0);
c516=cos(5.0*pai/16.0);
s516=sin(5.0*pai/16.0);
c716=cos(7.0*pai/16.0);
s716=sin(7.0*pai/16.0);
```

プログラミング(4)

■ 続き

```
for(i=0; i<4; i++) {  
    a[i] = data1[i] + data1[7-i];  
    a[7-i] = data1[i] - data1[7-i];  
}  
  
b[0] = a[0] + a[3];  
b[1] = a[1] + a[2];  
b[2] = a[1] - a[2];  
b[3] = a[0] - a[3];  
b[4] = a[4];  
b[5] = (a[6]-a[5])*c14;  
b[6] = (a[6]+a[5])*c14;  
b[7] = a[7];
```

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Programming(4)

■ Contd.

```
for(i=0; i<4; i++) {  
    a[i] = data1[i] + data1[7-i];  
    a[7-i] = data1[i] - data1[7-i];  
}  
  
b[0] = a[0] + a[3];  
b[1] = a[1] + a[2];  
b[2] = a[1] - a[2];  
b[3] = a[0] - a[3];  
b[4] = a[4];  
b[5] = (a[6]-a[5])*c14;  
b[6] = (a[6]+a[5])*c14;  
b[7] = a[7];
```

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プログラミング(5)

■ 続き

```
c[0] = (b[0]+b[1])*c14;  
c[1] = (b[0]-b[1])*c14;  
c[2] = b[2]*s18 + b[3]*c18;  
c[3] = -b[2]*s38 + b[3]*c38;  
c[4] = b[4] + b[5];  
c[5] = b[4] - b[5];  
c[6] = -b[6] + b[7];  
c[7] = b[6] + b[7];  
  
d[4] = c[4]*s116 + c[7]*c116;  
d[5] = c[5]*s516 + c[6]*c516;  
d[6] = -c[5]*s316 + c[6]*c316;  
d[7] = -c[4]*s716 + c[7]*c716;
```

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Programming(5)

■ Contd.

```
c[0] = (b[0]+b[1])*c14;  
c[1] = (b[0]-b[1])*c14;  
c[2] = b[2]*s18 + b[3]*c18;  
c[3] = -b[2]*s38 + b[3]*c38;  
c[4] = b[4] + b[5];  
c[5] = b[4] - b[5];  
c[6] = -b[6] + b[7];  
c[7] = b[6] + b[7];  
  
d[4] = c[4]*s116 + c[7]*c116;  
d[5] = c[5]*s516 + c[6]*c516;  
d[6] = -c[5]*s316 + c[6]*c316;  
d[7] = -c[4]*s716 + c[7]*c716;
```

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プログラミング(6)

■ 続き

```
data2[0] = c[0]/2.0;  
data2[2] = c[1]/2.0;  
data2[4] = c[2]/2.0;  
data2[6] = c[3]/2.0;  
data2[1] = d[4]/2.0;  
data2[5] = d[5]/2.0;  
data2[3] = d[6]/2.0;  
data2[7] = d[7]/2.0;  
}
```

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Programming(6)

■ Contd.

```
data2[0] = c[0]/2.0;  
data2[2] = c[1]/2.0;  
data2[4] = c[2]/2.0;  
data2[6] = c[3]/2.0;  
data2[1] = d[4]/2.0;  
data2[5] = d[5]/2.0;  
data2[3] = d[6]/2.0;  
data2[7] = d[7]/2.0;  
}
```

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対称波形のDFT

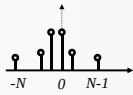
- 信号 $x(n)$ が原点に対して偶対称であるとき、サンプル点を1/2だけずらした区間 $(-N, N-1)$ のDFTは以下で与えられる

$$X(k) = \sum_{n=-N}^N x(n) \exp\left(\frac{-j2\pi k(n + 1/2)}{2N}\right)$$

- 信号の偶対称性から

$$x(-n-1) = x(n) \quad (n = 0, \dots, N-1)$$

が成り立つ



DFT for Symmetric Wave

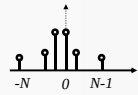
- When signal $x(n)$ is even-symmetric at 0, DFT of half-sample shifted signal in the range $(-N, N-1)$ is given by

$$X(k) = \sum_{n=-N}^N x(n) \exp\left(\frac{-j2\pi k(n + 1/2)}{2N}\right)$$

- It holds

$$x(-n-1) = x(n) \quad (n = 0, \dots, N-1)$$

by signal's symmetric nature



対称波形のDFT(2)

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-j2\pi k(n+1/2)}{2N}\right) \\ &\quad + \sum_{m=0}^{N-1} x(m) \exp\left(\frac{-j2\pi k((-m-1)+1/2)}{2N}\right) \\ &= \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-j2\pi k(n+1/2)}{2N}\right) \\ &\quad + \sum_{n=0}^{N-1} x(n) \exp\left(\frac{j2\pi k(n+1/2)}{2N}\right) \\ &= 2 \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \end{aligned}$$

← DCT II

DFT for Symmetric Wave(2)

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-j2\pi k(n+1/2)}{2N}\right) \\ &\quad + \sum_{m=0}^{N-1} x(m) \exp\left(\frac{-j2\pi k((-m-1)+1/2)}{2N}\right) \\ &= \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-j2\pi k(n+1/2)}{2N}\right) \\ &\quad + \sum_{n=0}^{N-1} x(n) \exp\left(\frac{j2\pi k(n+1/2)}{2N}\right) \\ &= 2 \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)k\pi}{2N}\right) \end{aligned}$$

← DCT II

対称波形のDFT(3)

- 原点に対して偶対称な波形のDFTはDCTに等しい
- 画像を相関という観点から眺めると、統計量は左右で変化しない

DFT for Symmetric Wave(3)

- DFT of signals which is even-symmetric at 0 equals to DCT
- Image can be viewed symmetric signal in the statistic sense

DCTの特徴

- KL変換に近い低域シーケンスへのパワー集中
- 直流成分は1個の係数で表現できる
- FFTに似たバタフライ演算による高速算法を実現可能
- 画像符号化 (DCT-II)、音響符号化 (M-DCT, MLT)
 - M-DCT: Modified DCT
 - LOT: Lapped Orthogonal Transform
 - MLT: Modulated LOT

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Characteristics of DCT

- Energy compaction performance to low sequency is close to KL Transform
- DC component can be represented by one coefficient
- Fast computational algorithm exists like FFT's butterfly computation
- Image coding (DCT-II)、Audio coding (M-DCT, MLT)
 - M-DCT: Modified DCT
 - LOT: Lapped Orthogonal Transform
 - MLT: Modulated LOT

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