信号理論

- No.5 フィルタ設計 -

渡辺　裕

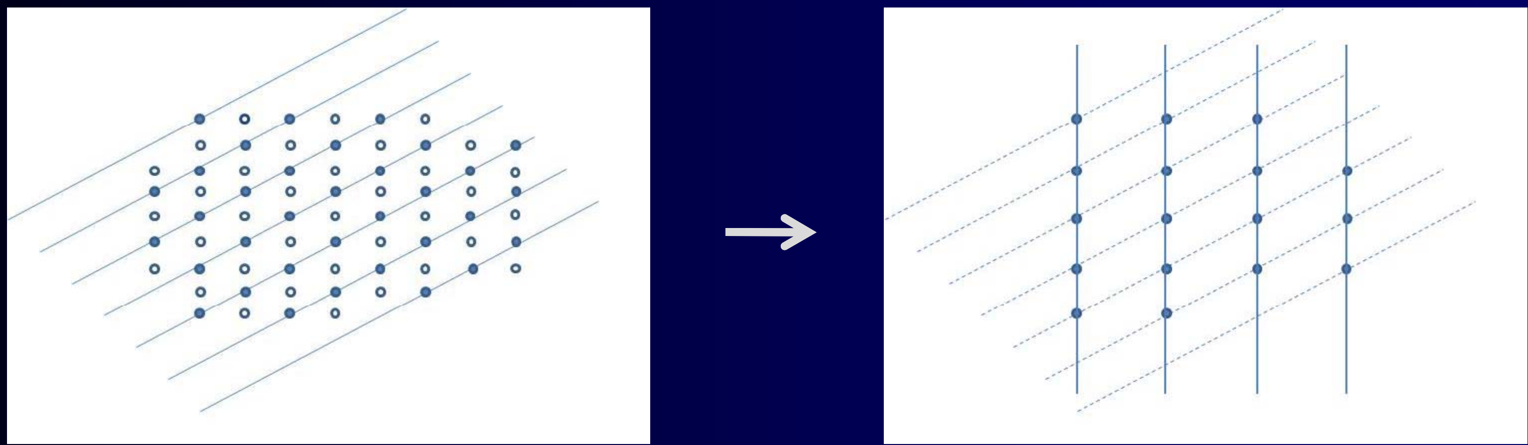
Signal Theory

- No.5 Filter Design -

Hiroshi Watanabe

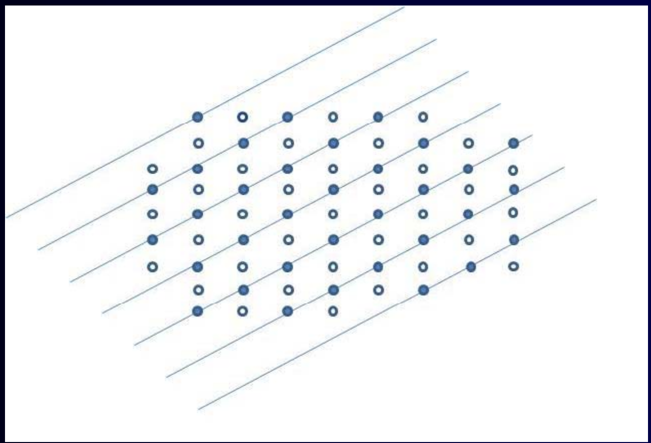
折返し歪 (1)

- サブサンプルによる空間領域での折り返し歪の例

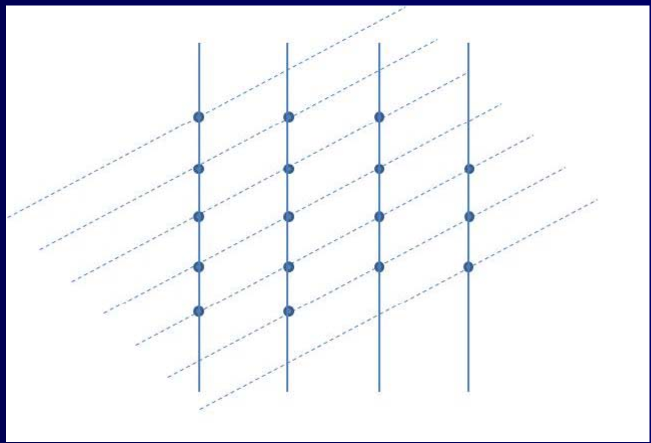


Aliasing Noise (1)

- Example of aliasing noise in a space domain by sub-sampling

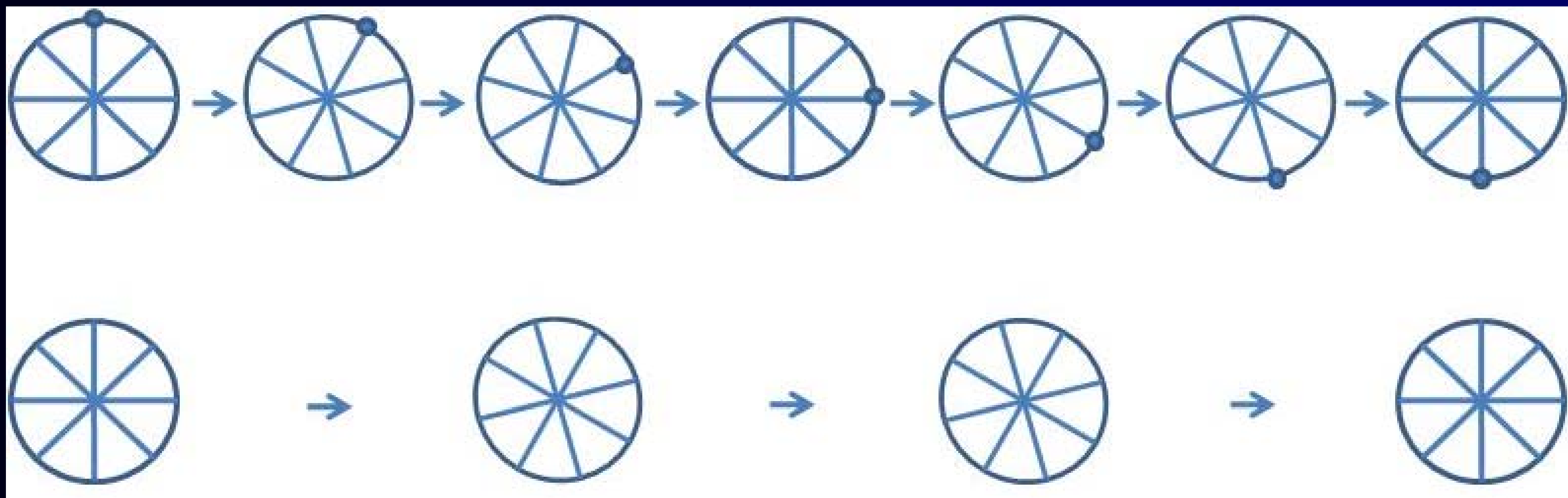


→



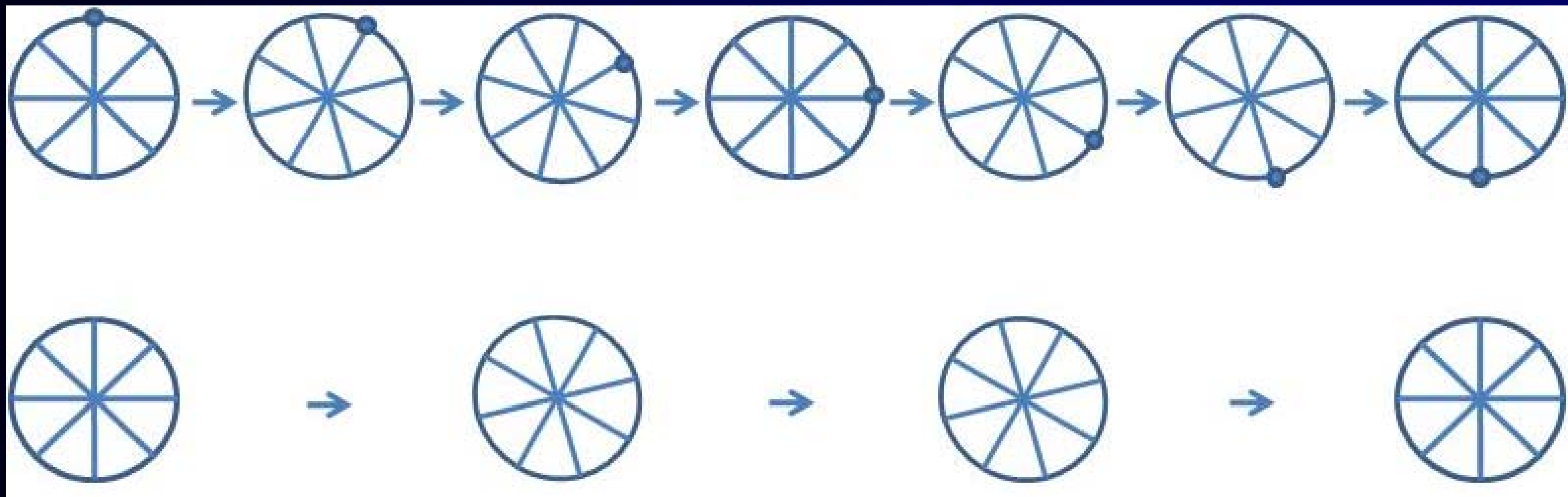
折返し歪 (2)

- フレーム間引きによる時間領域での折り返し歪の例



Aliasing Noise (2)

- Example of aliasing noise in a time domain by frame rate reduction



折り返し歪 (3)

- Bitmap file
  RGB 24 bit
  720x576 pels

Aliasing Noise (3)

- Bitmap file
  RGB 24 bit
  720x576 pels

折り返し歪 (4)

- 360x288 pels
  without LPF
  aliasing noise

Aliasing Noise (4)

- 360x288 pels
  without LPF
  aliasing noise

折り返し歪 (5)

- 360x288 pels
  with LPF
  2D-(1,2,1)

Aliasing Noise (5)

- 360x288 pels
  with LPF
  2D-(1,2,1)

解像度変換とディジタルフィルタ

- N:1のサブサンプリングの際には，通過帯域π/Nの低域通過型フィルタ (Low Pass Filter; LPF) を適用

- 伝達関数のz変換表現

$$H(z)=\sum_{k=0}^{N-1}h(k)z^{-k}$$

- 例

$$H(z)=\frac{1}{4}z^{1}+\frac{1}{2}z^{0}+\frac{1}{4}z^{-1}$$

Resolution Conversion and Digital Filter

- For N:1 sub-sampling, Low Pass Filter (LPF) with pass-band π/N

- Z-transform representation of transfer function

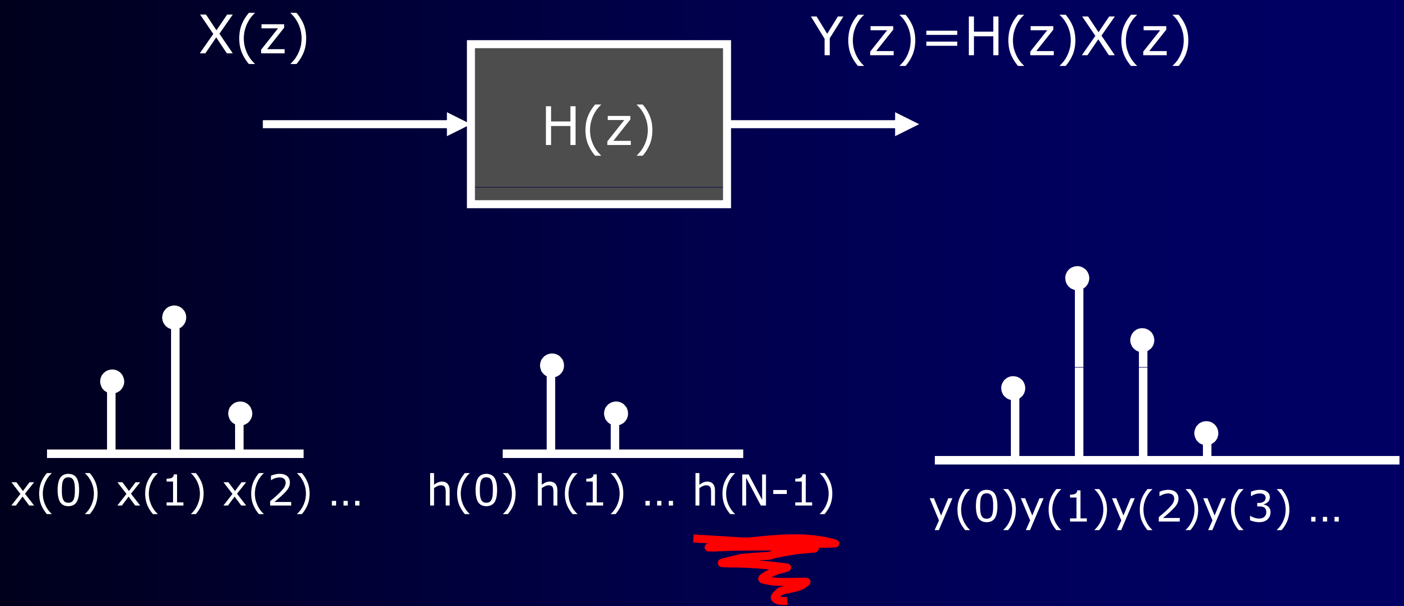
$$H(z) = \sum_{k=0}^{N-1} h(k) z^{-k}$$

- Example

$$H(z) = \frac{1}{4} z^{1} + \frac{1}{2} z^{0} + \frac{1}{4} z^{-1}$$

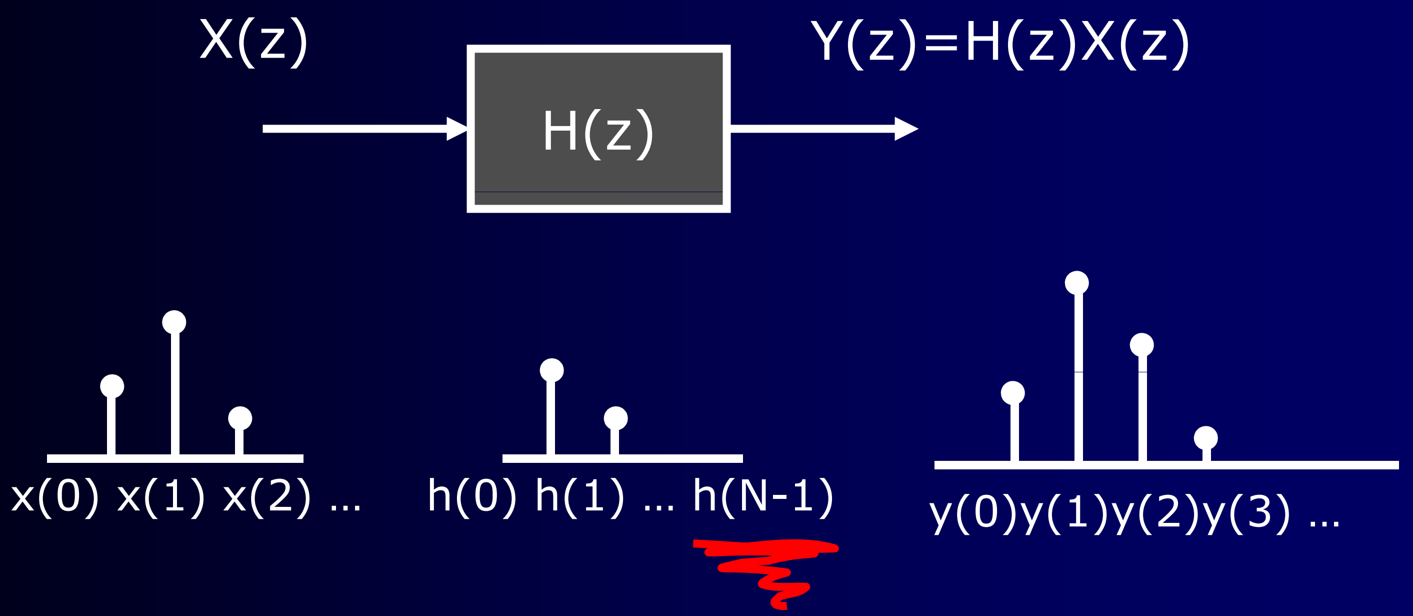
FIRフィルタ

- z-変換によるシステム記述



FIR Filter

- System description by z-transform



LPF(ローパスフィルタ)の設計

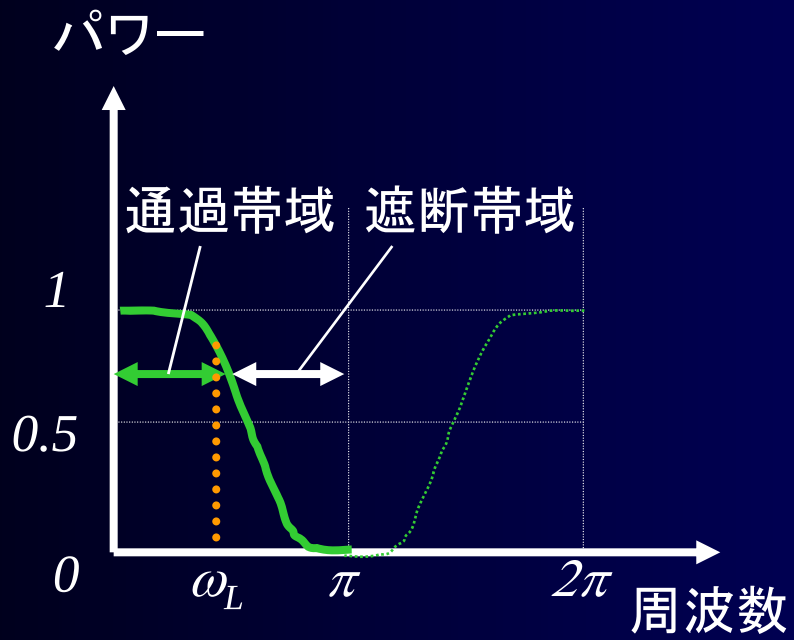
- FIRフィルタによるローパスフィルタ設計
  - 遮断周波数を決める
  - 伝達関数は理想ローパスフィルタを仮定する
  - フィルタ次数を決める
  - 係数を算出する
  - 係数と窓関数の積を求める

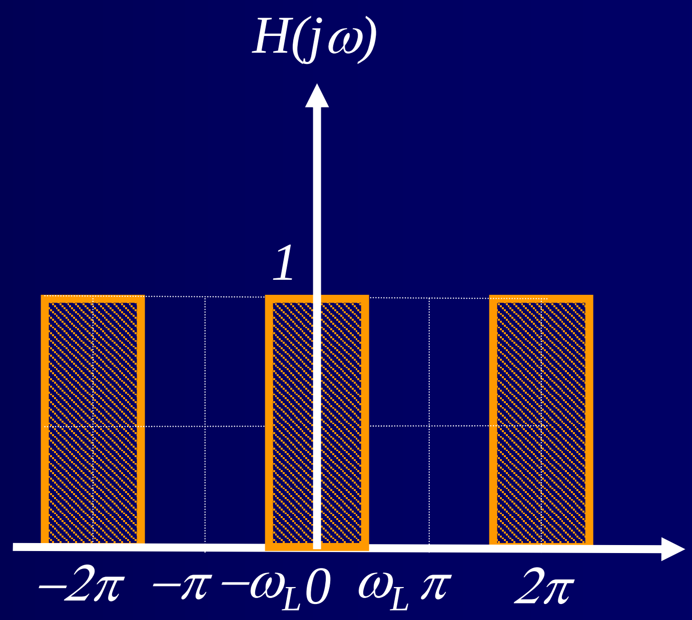
LPF Design

- Low-pass filter design by FIR filter
  - determine cut-off frequency
  - presume ideal low-pass filter as a transfer function
  - determine filter order
  - derive coefficients
  - multiply value of window function to coefficients

LPF遮断周波数

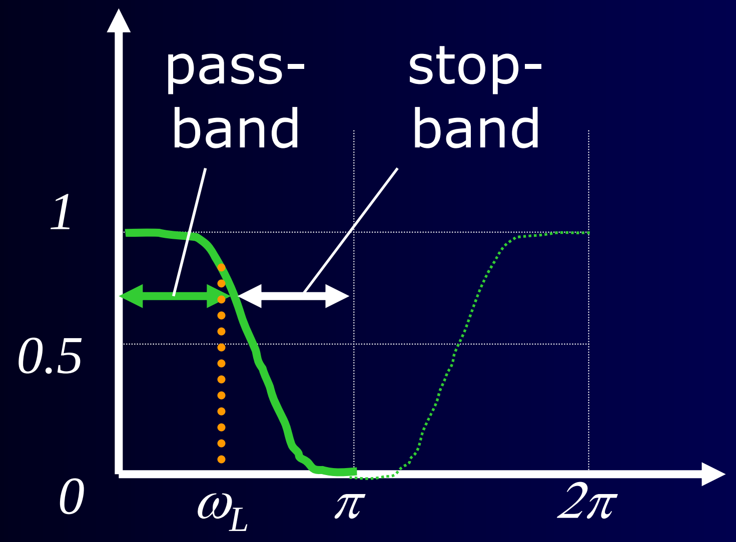
- 実際のローパスフィルタ特性と理想ローパスフィルタ特性
- 遮断周波数：ω_L

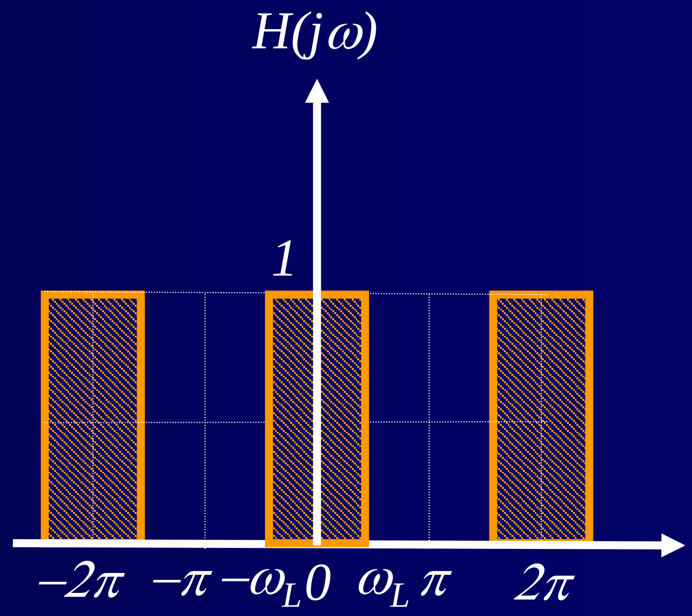




LPF Cut-off Frequency

- Actual and ideal low-pass filter characteristics
- cut-off frequency: ω_L





frequency

LPF伝達関数と係数の関係

- 伝達関数が

$$H(z)=\sum_{k=0}^{N-1}h(k)z^{-k}$$

のとき、係数 *h(k)* は次式で計算できる

$$h(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(j\omega)e^{jk\omega}d\omega \qquad (k=0,\cdots,N-1)$$

Relation Between LPF Transfer Function and Coefficient

- When the transfer function is given by

$$H(z) = \sum_{k=0}^{N-1} h(k) z^{-k}$$

coefficients $h(k)$ can be calculated by the following.

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \qquad (k = 0, \cdots, N-1)$$

LPF伝達関数と係数の関係 (2)

- なぜなら、

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{jk\omega}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{N-1} h(n)e^{-jn\omega}e^{jk\omega}d\omega \\
&= \frac{1}{2\pi}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} h(n)e^{-j(n-k)\omega}d\omega \\
&= \frac{1}{2\pi}h(k)\int_{-\pi}^{\pi} d\omega \\
&= \frac{1}{2\pi}h(k)\left[\omega\right]_{-\pi}^{\pi} \\
&= h(k)
\end{aligned}
$$

Relation Between LPF Transfer Function and Coefficient (2)

- Since,

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{jk\omega}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{N-1} h(n)e^{-jn\omega}e^{jk\omega}d\omega \\
&= \frac{1}{2\pi}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} h(n)e^{-j(n-k)\omega}d\omega \\
&= \frac{1}{2\pi}h(k)\int_{-\pi}^{\pi} d\omega \\
&= \frac{1}{2\pi}h(k)[\omega]_{-\pi}^{\pi} \\
&= h(k)
\end{aligned}
$$

復習

- [-π, π] における $exp(jn\omega)$ の積分

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{jn\omega} d\omega &= \int_{-\pi}^{\pi} (\cos n\omega + j \sin n\omega) d\omega \\
&= \left[\sin n\omega - j \cos n\omega\right]_{-\pi}^{\pi} \\
&= (\sin n\pi - j \cos n\pi) - (\sin n(-\pi) - j \cos n(-\pi)) \\
&= 0
\end{aligned}
$$

Review

- Integral of *exp(jnω)* at [*π, π*]

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{jn\omega} d\omega &= \int_{-\pi}^{\pi} (\cos n\omega + j \sin n\omega) d\omega \\
&= \left[\sin n\omega - j \cos n\omega\right]_{-\pi}^{\pi} \\
&= (\sin n\pi - j \cos n\pi) - (\sin n(-\pi) - j \cos n(-\pi)) \\
&= 0
\end{aligned}
$$

例

- $H(z)=1/4z^0+1/2z^{-1}+1/4z^{-2}$ の場合の係数導出

$$h(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}e^{0}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)d\omega=\frac{1}{4}$$

$$h(1)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}e^{0}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)e^{j\omega}d\omega=\frac{1}{2}$$

$$h(2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}e^{0}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)e^{j2\omega}d\omega=\frac{1}{4}$$

Example

- Derivation of coefficients for $H(z)=1/4z^0+1/2z^{-1}+1/4z^{-2}$

$$h(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}e^{0}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)d\omega=\frac{1}{4}$$

$$h(1)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}e^{0}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)e^{j\omega}d\omega=\frac{1}{2}$$

$$h(2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{4}e^{0}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)e^{j2\omega}d\omega=\frac{1}{4}$$

係数の算出

- 遮断周波数 ω_L, フィルタの次数 N の場合

$$h(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(j\omega)d\omega=\frac{1}{2\pi}\int_{-\omega_L}^{\omega_L}1d\omega=\frac{1}{2\pi}\left[\omega\right]_{-\omega_L}^{\omega_L}=\frac{\omega_L}{\pi}$$

$$h(1)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(j\omega)e^{j\omega}d\omega=\frac{1}{2\pi}\int_{-\omega_L}^{\omega_L}e^{j\omega}d\omega=\frac{1}{2\pi}\left[\frac{e^{j\omega}}{j}\right]_{-\omega_L}^{\omega_L}=\frac{\sin\omega_L}{\pi}$$

Derivation of Coefficients

- In case of cut-off frequency: ω_L, filter order N

$$h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)d\omega = \frac{1}{2\pi}\int_{-\omega_L}^{\omega_L} 1 d\omega = \frac{1}{2\pi}[\omega]_{-\omega_L}^{\omega_L} = \frac{\omega_L}{\pi}$$

$$h(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{j\omega}d\omega = \frac{1}{2\pi}\int_{-\omega_L}^{\omega_L} e^{j\omega}d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega}}{j}\right]_{-\omega_L}^{\omega_L} = \frac{\sin\omega_L}{\pi}$$

係数の算出 (2)

- 添字 k が負の場合

$$
\begin{aligned}
h(-1) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{-j\omega}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_L}^{\omega_L} e^{-j\omega}d\omega \\
&= \frac{1}{2\pi}\left[\frac{e^{-j\omega}}{-j}\right]_{-\omega_L}^{\omega_L} \\
&= \frac{\sin\omega_L}{\pi}
\end{aligned}
$$

Derivation of Coefficients (2)

- In case of negative sub-script k

$$h(-1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{-j\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_L}^{\omega_L} e^{-j\omega}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega}}{-j}\right]_{-\omega_L}^{\omega_L}$$

$$= \frac{\sin\omega_L}{\pi}$$

係数の算出 (3)

- 一般にLPFの場合，係数は次式で計算できる

$$h(0)=\frac{\omega_L}{\pi}$$

$$\begin{aligned}
h(k)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(j\omega)e^{jk\omega}d\omega\\
&=\frac{1}{2\pi}\int_{-\omega_L}^{\omega_L}e^{jk\omega}d\omega\\
&=\frac{1}{2\pi}\left[\frac{e^{jk\omega}}{jk}\right]_{-\omega_L}^{\omega_L}\\
&=\frac{\sin(k\omega_L)}{k\pi}\qquad (k=\pm1,\pm2,\cdots,\pm(N-1))
\end{aligned}$$

Derivation of Coefficients (3)

- In general, coefficients for LPF is given by the following.

$$h(0) = \frac{\omega_L}{\pi}$$

$$\begin{aligned} h(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{jk\omega}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_L}^{\omega_L} e^{jk\omega}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{jk\omega}}{jk}\right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin(k\omega_L)}{k\pi} \qquad (k = \pm 1, \pm 2, \cdots, \pm(N-1)) \end{aligned}$$

LPFの設計例

- 遮断周波数 $\omega_L = \pi/2$ を7次のフィルタで構成する場合

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

$$h(1) = h(-1) = \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin 3(\pi/2)}{3\pi} = \frac{-1}{3\pi} = -0.1061$$

Example of LPF Design

- Cut-off frequency $\omega_L = \pi/2$, 7-th order filter

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

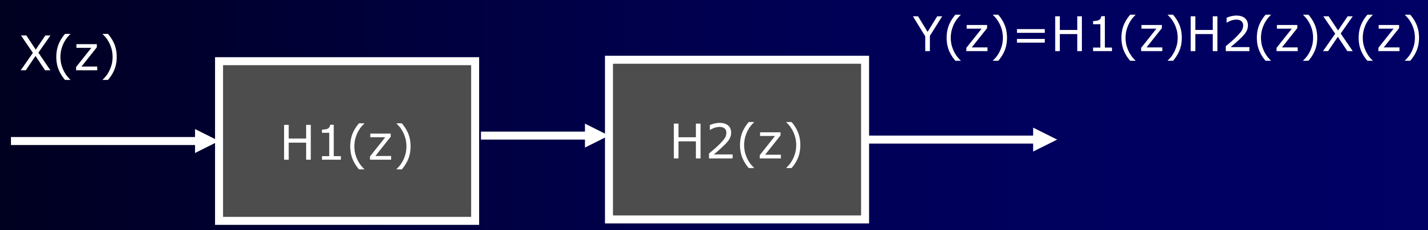
$$h(1) = h(-1) = \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin 3(\pi/2)}{3\pi} = \frac{-1}{3\pi} = -0.1061$$

遮断特性の改善

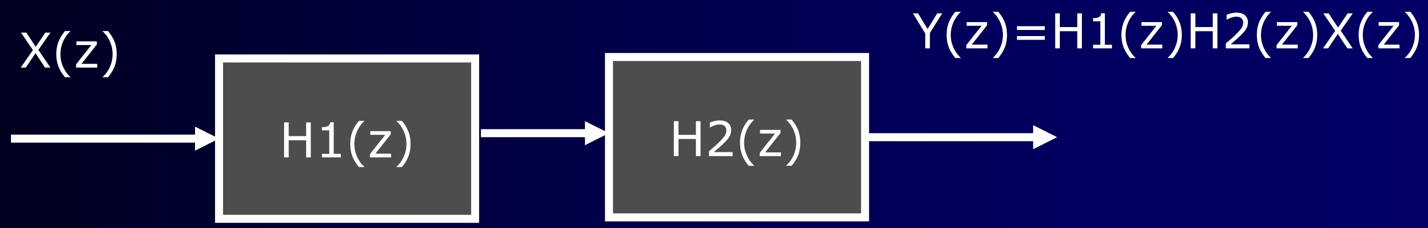
- ローパスフィルタの多段接続



- 窓関数
  - 有限次数のフィルタ構成．．．ギブスの現象によりリップルを生じる
  - 有限次数の打ち切りをなだらかに0に近づけるために，窓関数を使用する

Cut-off Characteristics Improvement

- Multi-stage connection of LPF



- Window function
  - Limited order filter: occurs ripple by Gibbs phenomena
  - Use window function to make both ends coefficients gradually close to zero

窓関数

- ハミング窓

$$w(k) = 0.54 + 0.46\cos\left(\frac{k\pi}{n}\right) \qquad (-n \le k \le n)$$

- ハニング窓

$$w(k) = 0.5\left(1 + \cos\left(\frac{k\pi}{n}\right)\right) \qquad (-n \le k \le n)$$

Window Function

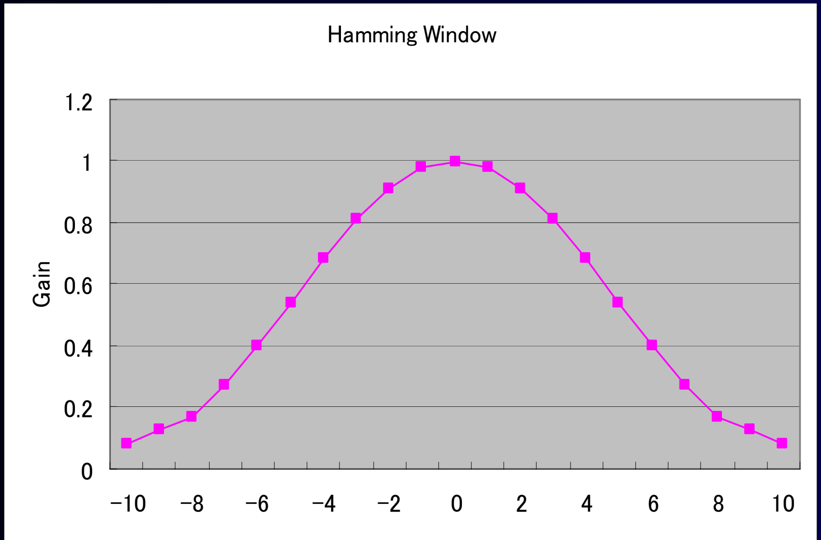
- Hamming Window

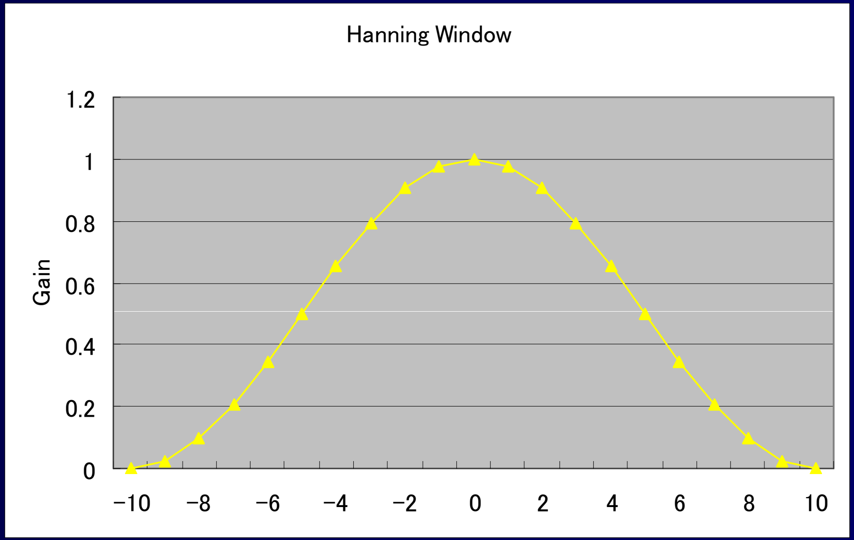
$$w(k) = 0.54 + 0.46\cos\left(\frac{k\pi}{n}\right) \qquad (-n \le k \le n)$$

- Hanning Window

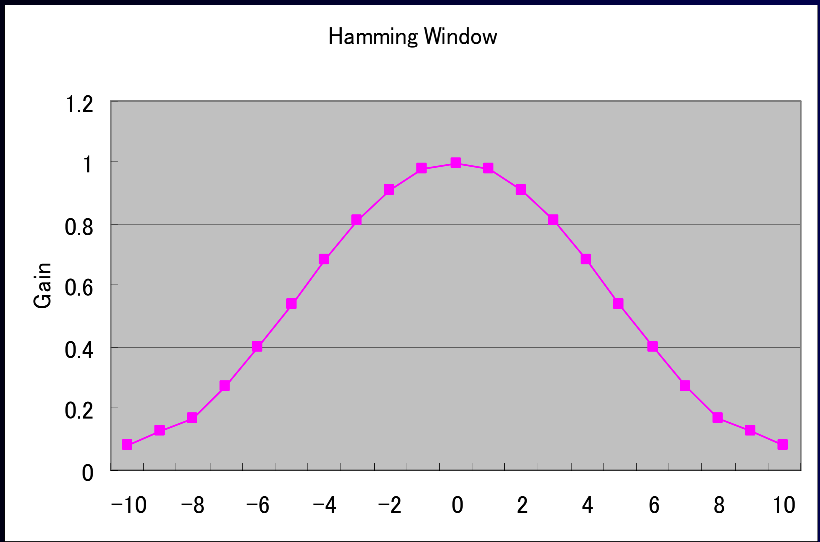
$$w(k) = 0.5\left(1 + \cos\left(\frac{k\pi}{n}\right)\right) \qquad (-n \le k \le n)$$

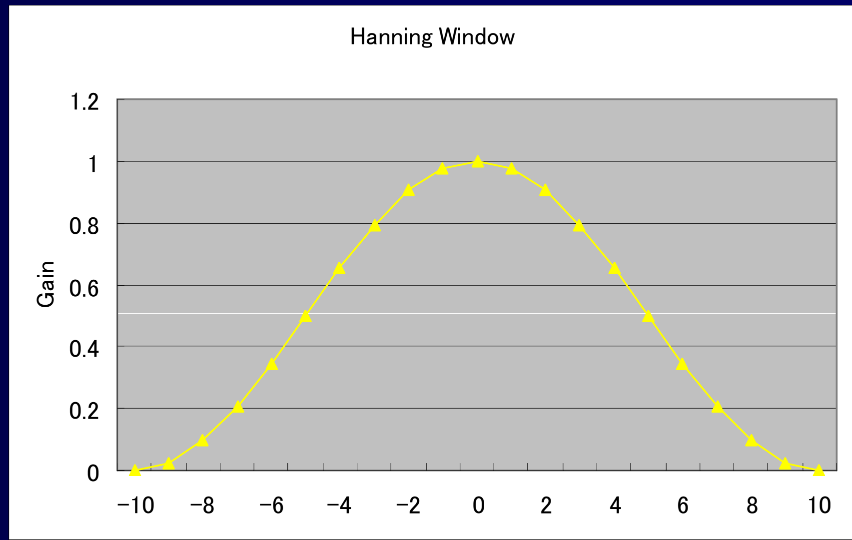
窓関数 (2)


Hamming Window
Gain
1.2
1
0.8
0.6
0.4
0.2
0
−10
−8
−6
−4
−2
0
2
4
6
8
10


Hanning Window
Gain
1.2
1
0.8
0.6
0.4
0.2
0
−10
−8
−6
−4
−2
0
2
4
6
8
10

Window Function (2)

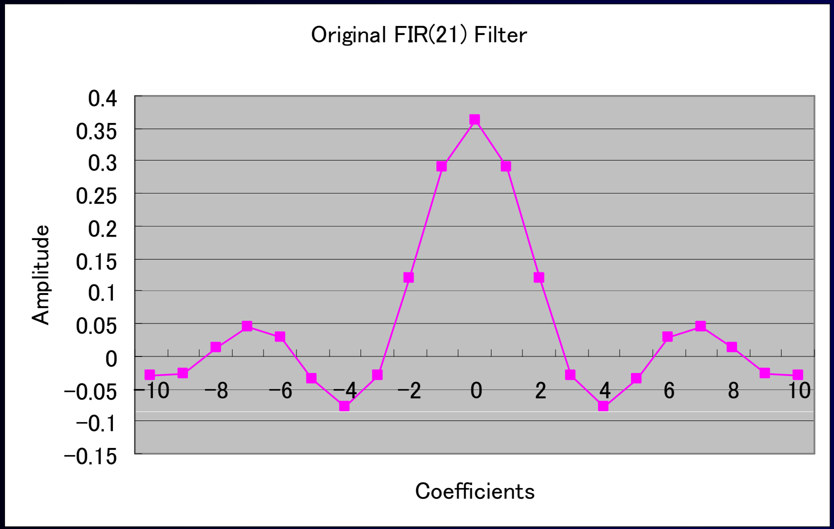


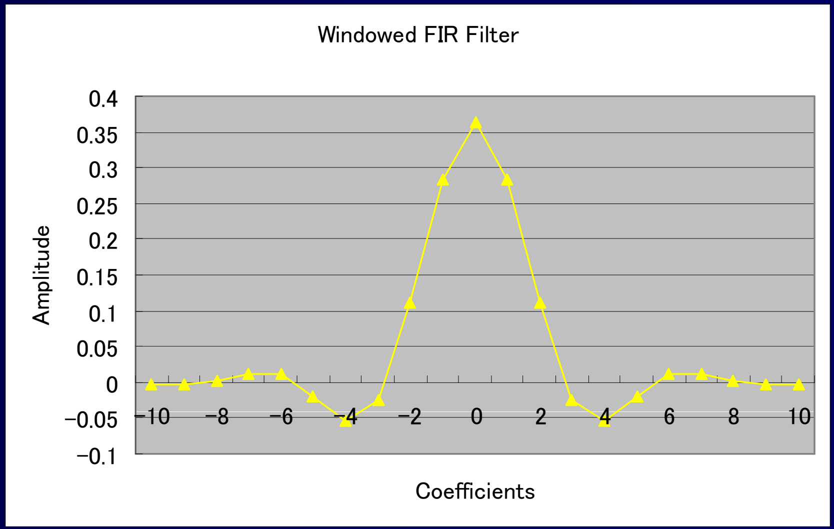


窓関数 (3)

- フィルタ係数と窓関数値の積を最終的なフィルタ係数とする

$$h_w(k) = h(k)w(k) \qquad (-n \le k \le n)$$

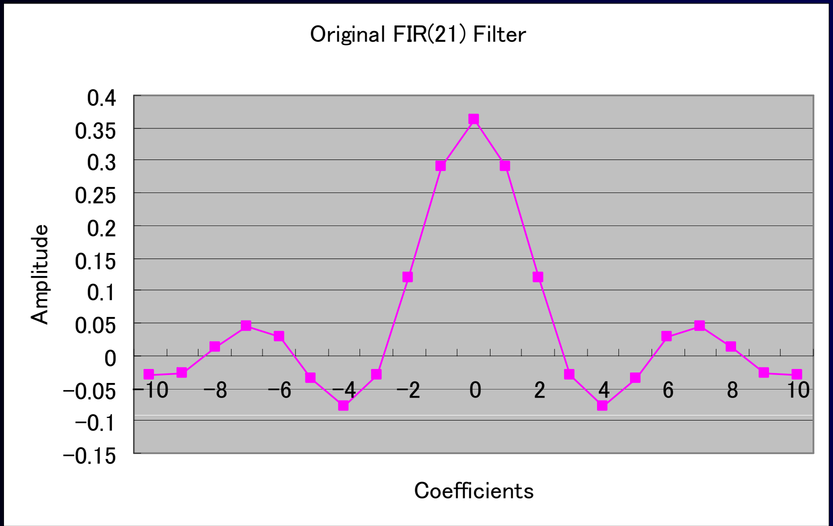


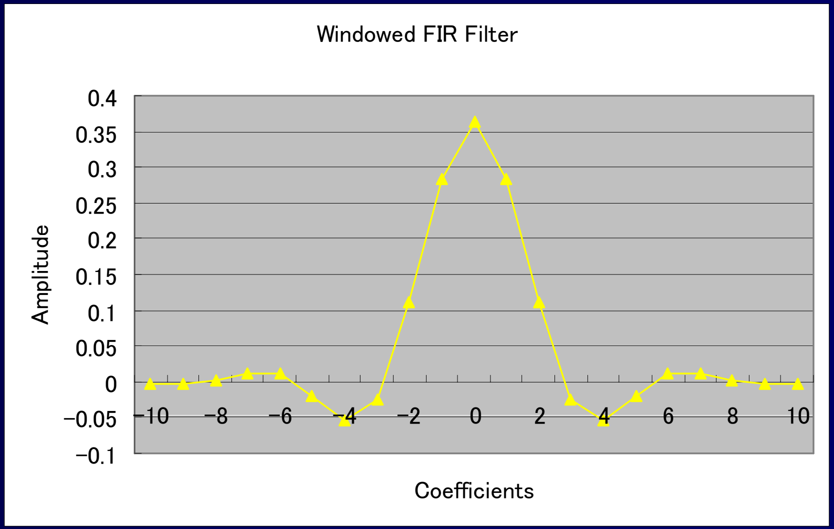


Window Fucntion (3)

- Filter coefficients are multiplied by values of window function

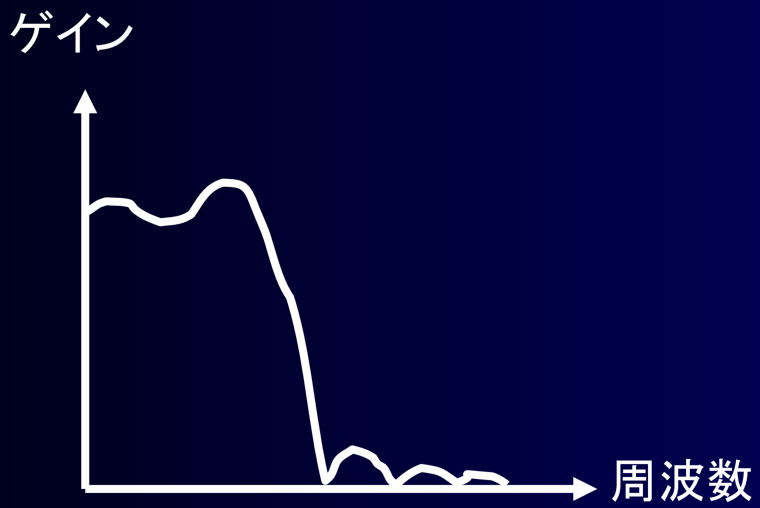
$$h_w(k) = h(k)w(k) \qquad (-n \le k \le n)$$



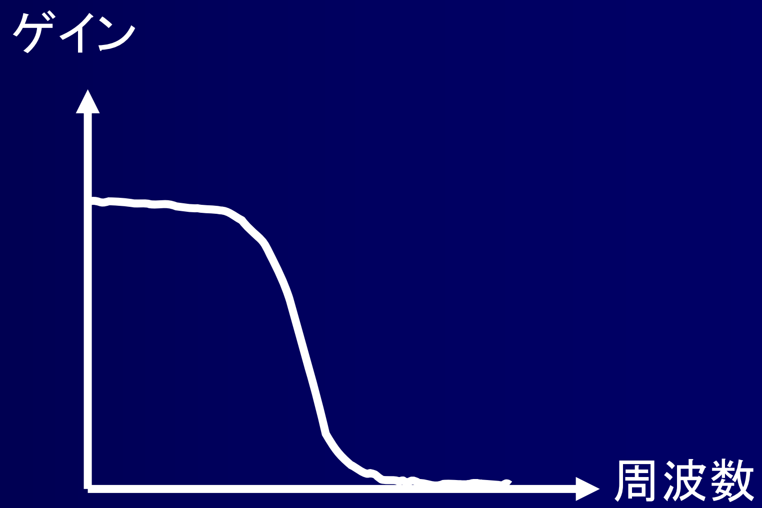


窓関数 (4)

■ 窓関数による周波数特性の変化



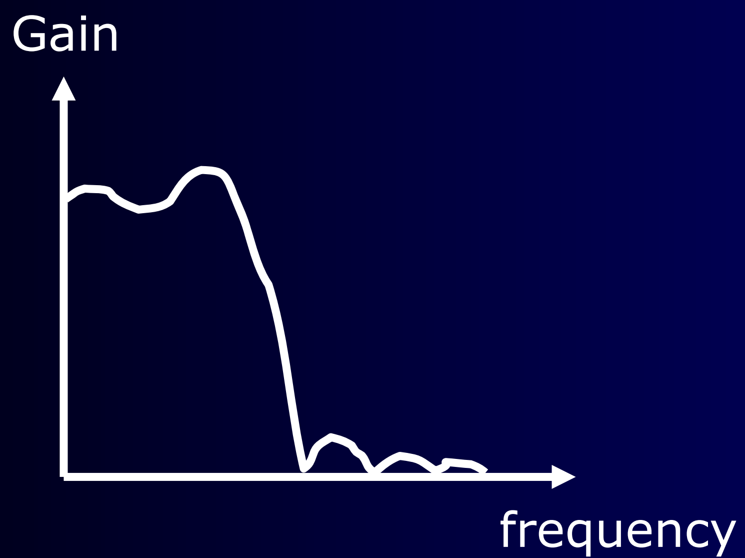
窓関数なし



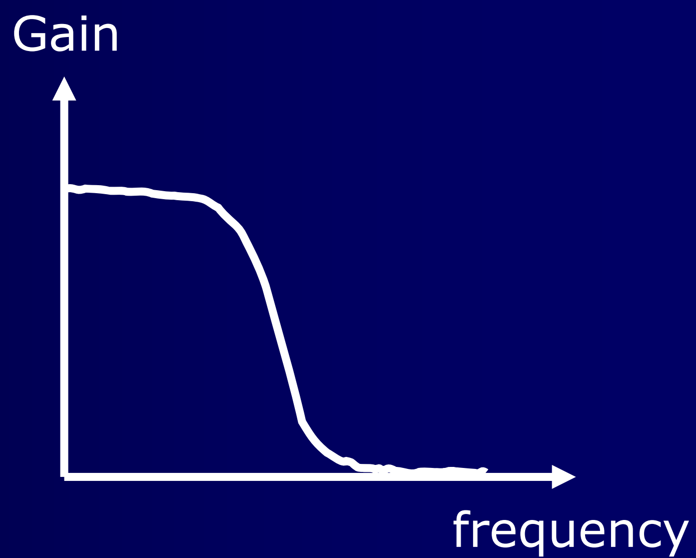
窓関数あり

Window Function (4)

- Change of frequency response by window function



without window function



with window function

HPF(ハイパスフィルタ)の設計

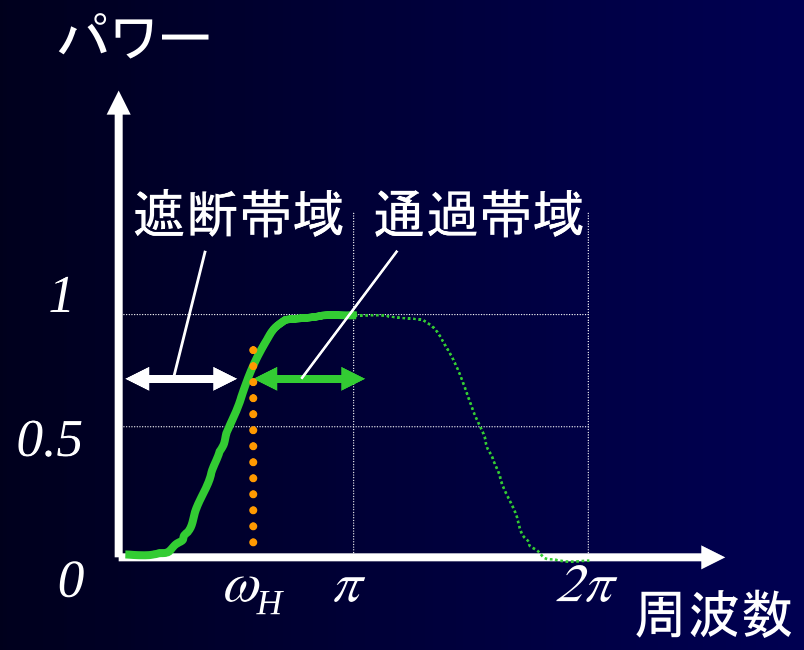
- FIRフィルタによるハイパスフィルタ設計
  - 遮断周波数を決める
  - 伝達関数は理想ハイパスフィルタを仮定する
  - フィルタ次数を決定する
  - ローパスフィルタの伝達関数をシフトして係数を算出する
  - 係数と窓関数の積を求める

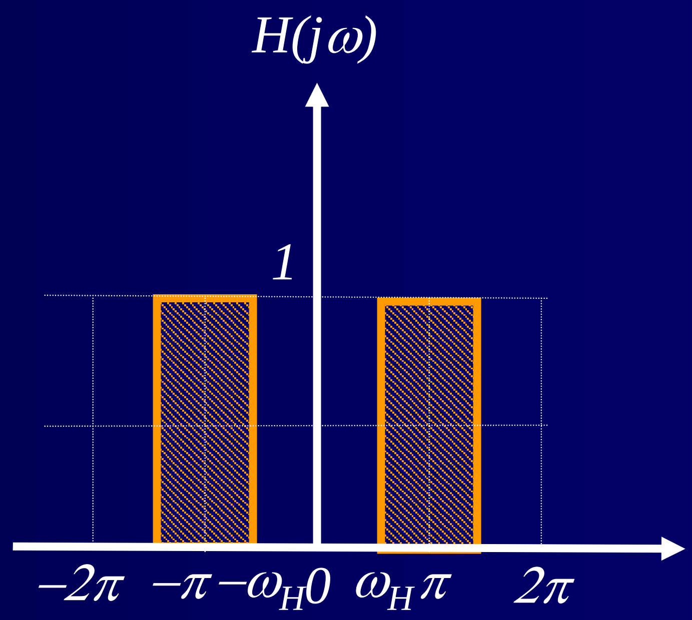
HPF Design

- High-pass filter design by FIR filter
  - Determine cut-off frequency
  - presume ideal high-pass filter as a transfer function
  - determine filter order
  - derive coefficients by shifting low-pass filter’s transfer function
  - multiply value of window function to coefficients

HPF遮断周波数

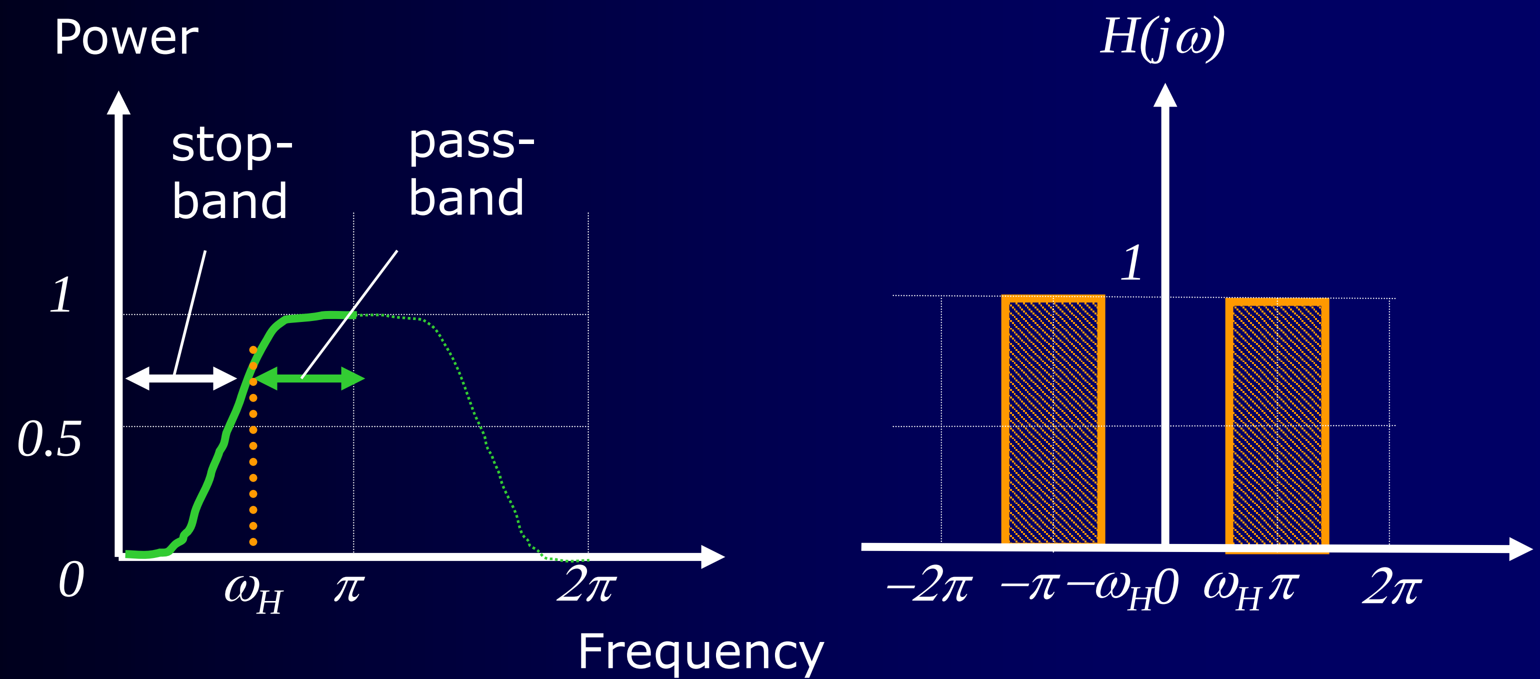
- 実際のハイパスフィルタ特性と理想ハイパスフィルタ特性
- 遮断周波数: $\omega_H = \pi - \omega_L$





HPF Cut-off Frequency

- Actual and ideal high-pass filter characteristics
- Cut-off frequency: $\omega_H = \pi - \omega_L$



HPF伝達関数と係数の関係

- ローパスフィルタの伝達関数が

$$H_L(z) = \sum_{k=-N}^{N} h(k) z^{-k}$$

のとき、ハイパスフィルタの伝達関数は，遮断周波数の関係

$$\omega_H = \pi - \omega_L$$

から　$z=exp(-jk\omega)$ において　ω を ω-π とおいたものに相当する

$$H_H(z) = \sum_{k=-N}^{N} h(k) e^{-jk(\omega-\pi)} = \sum_{k=-N}^{N} h(k) e^{-jk\omega} e^{jk\pi}$$

Relation Between HPF Transfer Function and Coefficients

- When LPF transfer function is given by

$$H_L(z) = \sum_{k=-N}^{N} h(k) z^{-k}$$

From the relation of cut-off frequency

$$\omega_H = \pi - \omega_L$$

HPF transfer function corresponds to the LPF's one replacing ω by ω-π at $z=exp(-jk\omega)$

$$H_H(z) = \sum_{k=-N}^{N} h(k) e^{-jk(\omega-\pi)} = \sum_{k=-N}^{N} h(k) e^{-jk\omega} e^{jk\pi}$$

HPF伝達関数と係数の関係 (2)

- 右辺の最後の項は

$$e^{jk\pi} = \cos k\pi + j \sin k\pi$$

であるから，k が偶数のときに 1，奇数のときに -1 となるから

$$H_H(j\omega) = (-1)^k H_L(j\omega)$$

となり，ハイパスフィルタの係数も $(-1)^k$ が乗じられた形となる

$$h_L(k) = (-1)^k \frac{\sin(k\omega_L)}{k\pi} \qquad (k = \pm 1, \pm 2, \cdots, \pm N)$$

ただし

$$h_L(0) = \omega_L / \pi$$

Relation Between HPF and Coefficients (2)

- Since

$$e^{jk\pi} = \cos k\pi + j \sin k\pi$$

the last term of the equation equals to *1* for even k, -1 for odd k

$$H_H(j\omega) = (-1)^k H_L(j\omega)$$

Thus, HPF coefficients are also given by multiplied form of $(-1)^k$

$$h_L(k) = (-1)^k \frac{\sin(k\omega_L)}{k\pi} \qquad (k = \pm 1, \pm 2, \cdots, \pm N)$$

where

$$h_L(0) = \omega_L / \pi$$

HPFの設計

- 遮断周波数 $\omega_H = \pi - \omega_L = \pi/2$ を7次のフィルタで構成する場合

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

$$h(1) = h(-1) = -\frac{\sin(\pi/2)}{\pi} = -\frac{1}{\pi} = -0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = -\frac{\sin 3(\pi/2)}{3\pi} = -\frac{-1}{3\pi} = 0.1061$$

HPF Design

- Cut-off frequency $\omega_H = \pi - \omega_L = \pi/2$, 7-th order filter

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

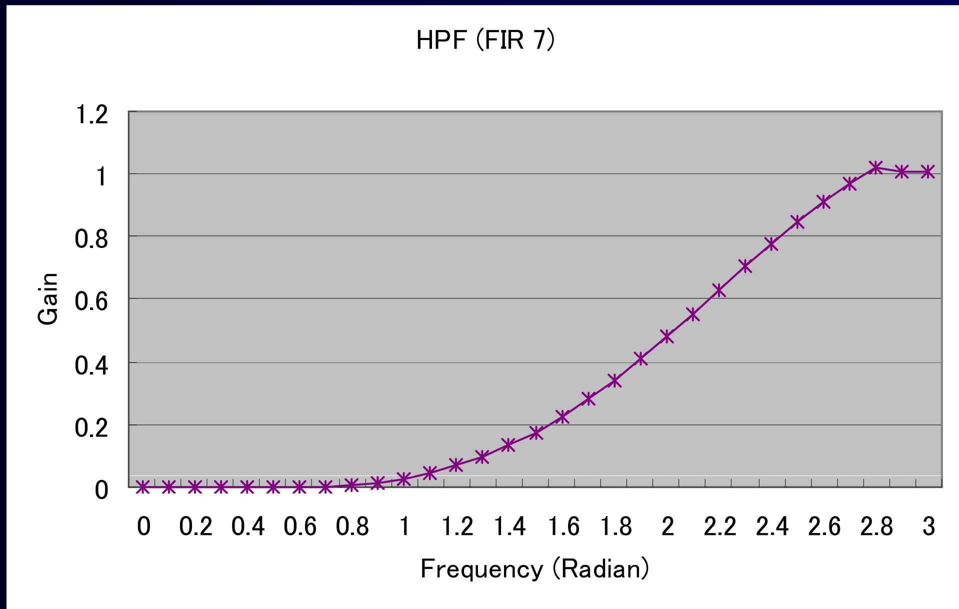
$$h(1) = h(-1) = -\frac{\sin(\pi/2)}{\pi} = -\frac{1}{\pi} = -0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = -\frac{\sin 3(\pi/2)}{3\pi} = -\frac{-1}{3\pi} = 0.1061$$

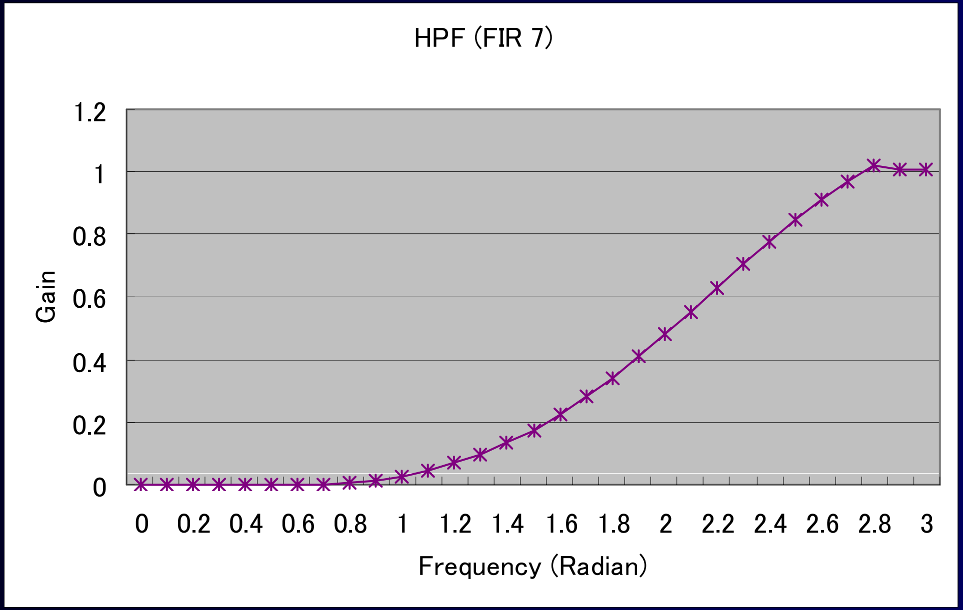
HPFの設計 (2)

- 周波数特性
  - ゲインを正規化していないことに注意
  - 本来はY軸はdBで表示



HPF Design (2)

- Frequency characteristics
  - Note that gain is not normalized
  - Gain is normally displayed by dB



係数近似

- 係数の分数近似
- 分母が2のべき乗

$$
\begin{aligned}
h(0) &= 0.5 \approx 8/16 \\
h(1) &= -0.3185 \approx -5/16 \\
h(2) &= 0 \\
h(3) &= 0.1061 \approx 1/16
\end{aligned}
$$

Coefficient Approximation

- approximate coefficients by fractional number
- denominator is approximated by two's power

$$h(0) = 0.5 \approx 8/16$$

$$h(1) = -0.3185 \approx -5/16$$

$$h(2) = 0$$

$$h(3) = 0.1061 \approx 1/16$$

問題

- 遮断周波数 $\omega_L = 3\pi/4$, 3次FIR型のLPFを設計せよ

Quiz

- Design 3rd order FIR filter having cut-off frequency $\omega_L = 3\pi/4$