

信号理論
- No.5 フィルタ設計 -

渡辺 裕

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Signal Theory
- No.5 Filter Design -

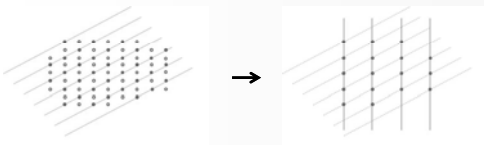
Hiroshi Watanabe

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折返し歪 (1)

- サブサンプリングによる空間領域での折返し歪の例

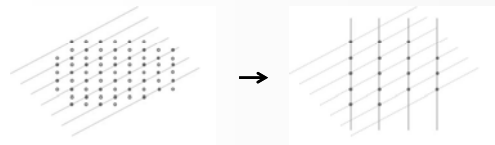


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Aliasing Noise (1)

- Example of aliasing noise in a space domain by sub-sampling

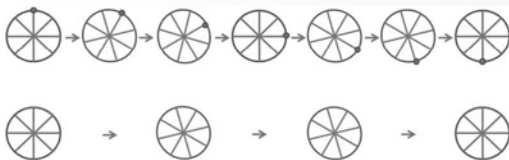


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折返し歪 (2)

- フレーム間引きによる時間領域での折返し歪の例

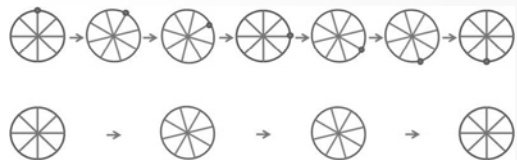


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Aliasing Noise (2)

- Example of aliasing noise in a time domain by frame rate reduction



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折り返し歪 (3)

- Bitmap file
RGB 24 bit
720x576 pels



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Aliasing Noise (3)

- Bitmap file
RGB 24 bit
720x576 pels



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折り返し歪 (4)

- 360x288 pels
without LPF
aliasing noise



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Aliasing Noise (4)

- 360x288 pels
without LPF
aliasing noise



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折り返し歪 (5)

- 360x288 pels
with LPF
2D-(1,2,1)



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Aliasing Noise (5)

- 360x288 pels
with LPF
2D-(1,2,1)



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解像度変換とデジタルフィルタ

- N:1のサブサンプリングの際には、通過帯域 π/N の低域通過型フィルタ (Low Pass Filter; LPF) を適用

- 伝達関数のz変換表現

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

- 例

$$H(z) = \frac{1}{4}z^1 + \frac{1}{2}z^0 + \frac{1}{4}z^{-1}$$

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Resolution Conversion and Digital Filter

- For N:1 sub-sampling, Low Pass Filter (LPF) with pass-band π/N

- Z-transform representation of transfer function

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

- Example

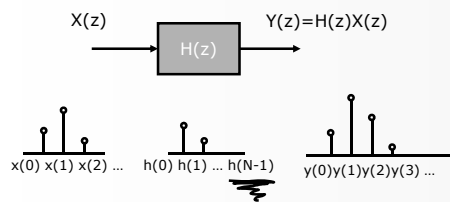
$$H(z) = \frac{1}{4}z^1 + \frac{1}{2}z^0 + \frac{1}{4}z^{-1}$$

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FIRフィルタ

- z-変換によるシステム記述

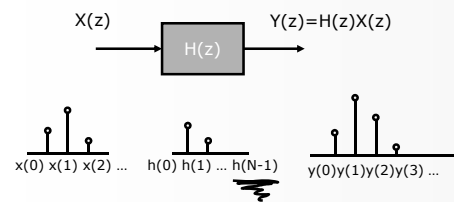


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FIR Filter

- System description by z-transform



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LPF(ローパスフィルタ)の設計

- FIRフィルタによるローパスフィルタ設計
 - 遮断周波数を定める
 - 伝達関数は理想ローパスフィルタを仮定する
 - フィルタ次数を決める
 - 係数を算出する
 - 係数と窓関数の積を求める

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LPF Design

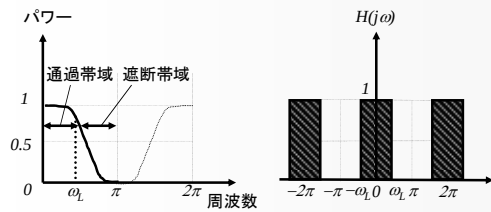
- Low-pass filter design by FIR filter
 - determine cut-off frequency
 - presume ideal low-pass filter as a transfer function
 - determine filter order
 - derive coefficients
 - multiply value of window function to coefficients

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LPF遮断周波数

- 実際のローパスフィルタ特性と理想ローパスフィルタ特性
- 遮断周波数: ω_L

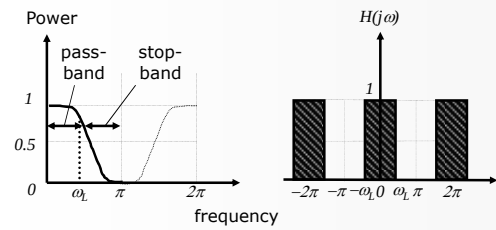


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LPF Cut-off Frequency

- Actual and ideal low-pass filter characteristics
- cut-off frequency: ω_L



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LPF伝達関数と係数の関係

- 伝達関数が

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

のとき、係数 $h(k)$ は次式で計算できる

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \quad (k = 0, \dots, N-1)$$

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Relation Between LPF Transfer Function and Coefficient

- When the transfer function is given by

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

coefficients $h(k)$ can be calculated by the following.

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \quad (k = 0, \dots, N-1)$$

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LPF伝達関数と係数の関係 (2)

- なぜなら、

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{N-1} h(n) e^{-jn\omega} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \sum_{n=0}^{N-1} h(n) \int_{-\pi}^{\pi} e^{-j(n-k)\omega} d\omega \\ &= \frac{1}{2\pi} h(k) \int_{-\pi}^{\pi} d\omega \\ &= \frac{1}{2\pi} h(k) [\omega]_{-\pi}^{\pi} \\ &= h(k) \end{aligned}$$

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Relation Between LPF Transfer Function and Coefficient (2)

- Since,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{N-1} h(n) e^{-jn\omega} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \sum_{n=0}^{N-1} h(n) \int_{-\pi}^{\pi} e^{-j(n-k)\omega} d\omega \\ &= \frac{1}{2\pi} h(k) \int_{-\pi}^{\pi} d\omega \\ &= \frac{1}{2\pi} h(k) [\omega]_{-\pi}^{\pi} \\ &= h(k) \end{aligned}$$

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復習

- $[-\pi, \pi]$ における $\exp(jn\omega)$ の積分

$$\begin{aligned}\int_{-\pi}^{\pi} e^{jn\omega} d\omega &= \int_{-\pi}^{\pi} (\cos n\omega + j \sin n\omega) d\omega \\ &= [\sin n\omega - j \cos n\omega]_{-\pi}^{\pi} \\ &= (\sin n\pi - j \cos n\pi) - (\sin n(-\pi) - j \cos n(-\pi)) \\ &= 0\end{aligned}$$

Review

- Integral of $\exp(jn\omega)$ at $[-\pi, \pi]$

$$\begin{aligned}\int_{-\pi}^{\pi} e^{jn\omega} d\omega &= \int_{-\pi}^{\pi} (\cos n\omega + j \sin n\omega) d\omega \\ &= [\sin n\omega - j \cos n\omega]_{-\pi}^{\pi} \\ &= (\sin n\pi - j \cos n\pi) - (\sin n(-\pi) - j \cos n(-\pi)) \\ &= 0\end{aligned}$$

例

- $H(z) = 1/4z^0 + 1/2z^{-1} + 1/4z^{-2}$ の場合の係数導出

$$h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) d\omega = \frac{1}{4}$$

$$h(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j\omega} d\omega = \frac{1}{2}$$

$$h(2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j2\omega} d\omega = \frac{1}{4}$$

Example

- Derivation of coefficients for $H(z) = 1/4z^0 + 1/2z^{-1} + 1/4z^{-2}$

$$h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) d\omega = \frac{1}{4}$$

$$h(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j\omega} d\omega = \frac{1}{2}$$

$$h(2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j2\omega} d\omega = \frac{1}{4}$$

係数の算出

- 遮断周波数 ω_L , フィルタの次数 N の場合

$$\begin{aligned}h(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} 1 d\omega \\ &= \frac{1}{2\pi} [\omega]_{-\omega_L}^{\omega_L} \\ &= \frac{\omega_L}{\pi}\end{aligned}$$

$$\begin{aligned}h(1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{j\omega}}{j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi}\end{aligned}$$

Derivation of Coefficients

- In case of cut-off frequency: ω_L , filter order N

$$\begin{aligned}h(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} 1 d\omega \\ &= \frac{1}{2\pi} [\omega]_{-\omega_L}^{\omega_L} \\ &= \frac{\omega_L}{\pi}\end{aligned}$$

$$\begin{aligned}h(1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{j\omega}}{j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi}\end{aligned}$$

係数の算出 (2)

- 添字 k が負の場合

$$\begin{aligned} h(-1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{-j\omega}}{-j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi} \end{aligned}$$

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Derivation of Coefficients (2)

- In case of negative sub-script k

$$\begin{aligned} h(-1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{-j\omega}}{-j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi} \end{aligned}$$

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係数の算出 (3)

- 一般にLPFの場合、係数は次式で計算できる

$$\begin{aligned} h(0) &= \frac{\omega_L}{\pi} \\ h(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{jk\omega}}{jk} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm(N-1)) \end{aligned}$$

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Derivation of Coefficients (3)

- In general, coefficients for LPF is given by the following.

$$\begin{aligned} h(0) &= \frac{\omega_L}{\pi} \\ h(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{jk\omega}}{jk} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm(N-1)) \end{aligned}$$

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LPFの設計例

- 遮断周波数 $\omega_L = \pi/2$ を7次のフィルタで構成する場合

$$\begin{aligned} h(0) &= \frac{\pi/2}{\pi} = 0.5 \\ h(1) = h(-1) &= \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3185 \\ h(2) = h(-2) &= \frac{\sin 2(\pi/2)}{2\pi} = 0 \\ h(3) = h(-3) &= \frac{\sin 3(\pi/2)}{3\pi} = \frac{-1}{3\pi} = -0.1061 \end{aligned}$$

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Example of LPF Design

- Cut-off frequency $\omega_L = \pi/2$, 7-th order filter

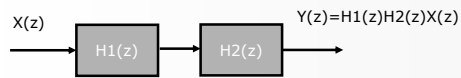
$$\begin{aligned} h(0) &= \frac{\pi/2}{\pi} = 0.5 \\ h(1) = h(-1) &= \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3185 \\ h(2) = h(-2) &= \frac{\sin 2(\pi/2)}{2\pi} = 0 \\ h(3) = h(-3) &= \frac{\sin 3(\pi/2)}{3\pi} = \frac{-1}{3\pi} = -0.1061 \end{aligned}$$

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遮断特性の改善

■ ローパスフィルタの多段接続



■ 窓関数

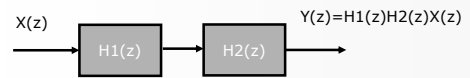
- 有限次数のフィルタ構成... ギブスの現象によりリップルを生じる
- 有限次数の打ち切りをなだらかに0に近づけるために、窓関数を使用する

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Cut-off Characteristics Improvement

■ Multi-stage connection of LPF



■ Window function

- Limited order filter: occurs ripple by Gibbs phenomena
- Use window function to make both ends coefficients gradually close to zero

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窓関数

■ ハミング窓

$$w(k) = 0.54 + 0.46 \cos\left(\frac{k\pi}{n}\right) \quad (-n \leq k \leq n)$$

■ ハニング窓

$$w(k) = 0.5 \left(1 + \cos\left(\frac{k\pi}{n}\right) \right) \quad (-n \leq k \leq n)$$

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Window Function

■ Hamming Window

$$w(k) = 0.54 + 0.46 \cos\left(\frac{k\pi}{n}\right) \quad (-n \leq k \leq n)$$

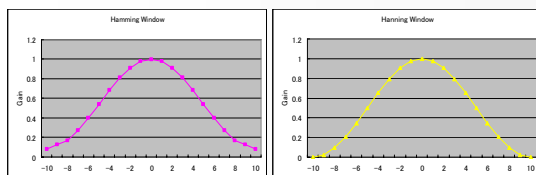
■ Hanning Window

$$w(k) = 0.5 \left(1 + \cos\left(\frac{k\pi}{n}\right) \right) \quad (-n \leq k \leq n)$$

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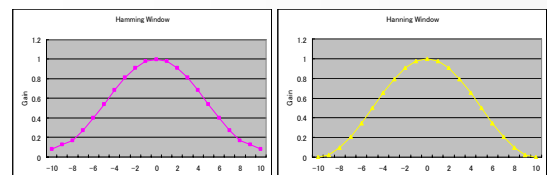
窓関数 (2)



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Window Function (2)



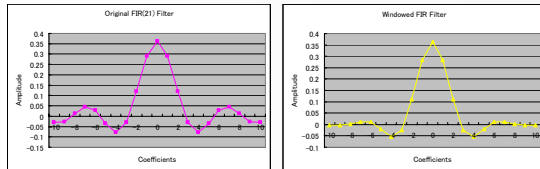
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窓関数 (3)

- フィルタ係数と窓関数値の積を最終的なフィルタ係数とする

$$h_w(k) = h(k)w(k) \quad (-n \leq k \leq n)$$



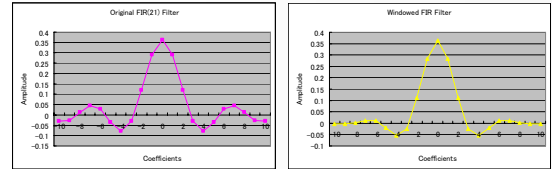
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Window Function (3)

- Filter coefficients are multiplied by values of window function

$$h_w(k) = h(k)w(k) \quad (-n \leq k \leq n)$$

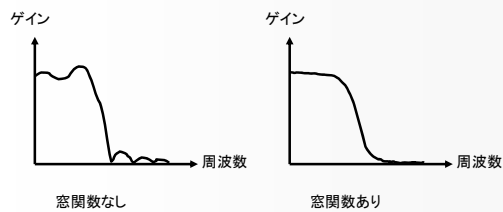


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窓関数 (4)

- 窓関数による周波数特性の変化

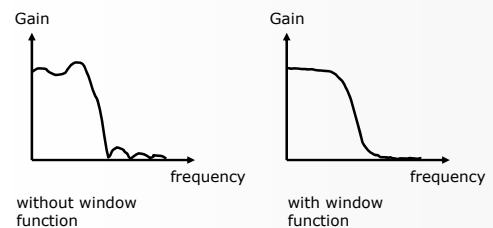


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Window Function (4)

- Change of frequency response by window function



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HPF(ハイパスフィルタ)の設計

- FIRフィルタによるハイパスフィルタ設計
 - 遮断周波数を決める
 - 伝達関数は理想ハイパスフィルタを仮定する
 - フィルタ次数を決定する
 - ローパスフィルタの伝達関数をシフトして係数を算出する
 - 係数と窓関数の積を求める

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HPF Design

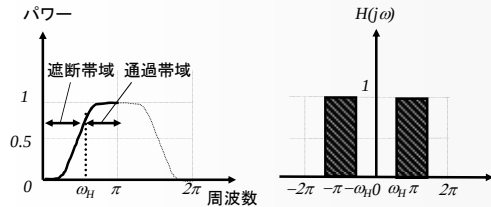
- High-pass filter design by FIR filter
 - Determine cut-off frequency
 - presume ideal high-pass filter as a transfer function
 - determine filter order
 - derive coefficients by shifting low-pass filter's transfer function
 - multiply value of window function to coefficients

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HPF遮断周波数

- 実際のハイパスフィルタ特性と理想ハイパスフィルタ特性
- 遮断周波数: $\omega_H = \pi - \omega_L$

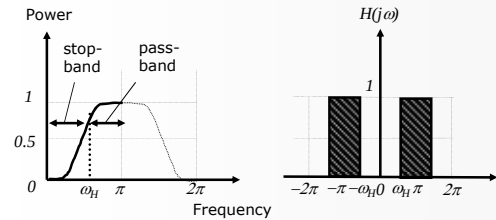


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HPF Cut-off Frequency

- Actual and ideal high-pass filter characteristics
- Cut-off frequency: $\omega_H = \pi - \omega_L$



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HPF伝達関数と係数の関係

- ローパスフィルタの伝達関数が

$$H_L(z) = \sum_{k=-N}^N h(k)z^{-k}$$

のとき、ハイパスフィルタの伝達関数は、遮断周波数の関係

$$\omega_H = \pi - \omega_L$$

から $z = \exp(-jk\omega)$ において ω を $\omega - \pi$ とおいたものに相当する

$$H_H(z) = \sum_{k=-N}^N h(k)e^{-jk(\omega-\pi)} = \sum_{k=-N}^N h(k)e^{-jk\omega} e^{jk\pi}$$

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Relation Between HPF Transfer Function and Coefficients

- When LPF transfer function is given by

$$H_L(z) = \sum_{k=-N}^N h(k)z^{-k}$$

From the relation of cut-off frequency

$$\omega_H = \pi - \omega_L$$

HPF transfer function corresponds to the LPF's one replacing ω by $\omega - \pi$ at $z = \exp(-jk\omega)$

$$H_H(z) = \sum_{k=-N}^N h(k)e^{-jk(\omega-\pi)} = \sum_{k=-N}^N h(k)e^{-jk\omega} e^{jk\pi}$$

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HPF伝達関数と係数の関係 (2)

- 右辺の最後の項は

$$e^{jk\pi} = \cos k\pi + j \sin k\pi$$

であるから、 k が偶数のときに 1, 奇数のときに -1 となるから

$$H_H(j\omega) = (-1)^k H_L(j\omega)$$

となり、ハイパスフィルタの係数も $(-1)^k$ が乗じられた形となる

$$h_L(k) = (-1)^k \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm N)$$

ただし

$$h_L(0) = \frac{\omega_L}{\pi}$$

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Relation Between HPF and Coefficients (2)

- Since

$$e^{jk\pi} = \cos k\pi + j \sin k\pi$$

the last term of the equation equals to 1 for even k , -1 for odd k

$$H_H(j\omega) = (-1)^k H_L(j\omega)$$

Thus, HPF coefficients are also given by multiplied form of $(-1)^k$

$$h_L(k) = (-1)^k \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm N)$$

where

$$h_L(0) = \frac{\omega_L}{\pi}$$

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HPFの設計

- 遮断周波数 $\omega_H = \pi$ - $\omega_L = \pi/2$ を7次のフィルタで構成する場合

$$\begin{aligned} h(0) &= \frac{\pi/2}{\pi} = 0.5 \\ h(1) &= h(-1) = -\frac{\sin(\pi/2)}{\pi} = -\frac{1}{\pi} = -0.3185 \\ h(2) &= h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0 \\ h(3) &= h(-3) = -\frac{\sin 3(\pi/2)}{3\pi} = -\frac{-1}{3\pi} = 0.1061 \end{aligned}$$

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HPF Design

- Cut-off frequency $\omega_H = \pi$ - $\omega_L = \pi/2$, 7-th order filter

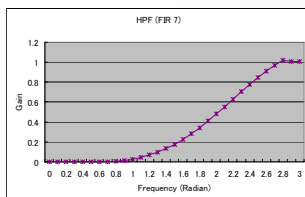
$$\begin{aligned} h(0) &= \frac{\pi/2}{\pi} = 0.5 \\ h(1) &= h(-1) = -\frac{\sin(\pi/2)}{\pi} = -\frac{1}{\pi} = -0.3185 \\ h(2) &= h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0 \\ h(3) &= h(-3) = -\frac{\sin 3(\pi/2)}{3\pi} = -\frac{-1}{3\pi} = 0.1061 \end{aligned}$$

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HPFの設計 (2)

- 周波数特性
 - ゲインを正規化していないことに注意
 - 本来はY軸はdBで表示

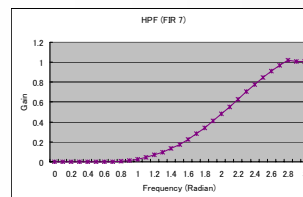


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HPF Design (2)

- Frequency characteristics
 - Note that gain is not normalized
 - Gain is normally displayed by dB



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係数近似

- 係数の分数近似
- 分母が2のべき乗

$$\begin{aligned} h(0) &= 0.5 \approx 8/16 \\ h(1) &= -0.3185 \approx -5/16 \\ h(2) &= 0 \\ h(3) &= 0.1061 \approx 1/16 \end{aligned}$$

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Coefficient Approximation

- approximate coefficients by fractional number
- denominator is approximated by two's power

$$\begin{aligned} h(0) &= 0.5 \approx 8/16 \\ h(1) &= -0.3185 \approx -5/16 \\ h(2) &= 0 \\ h(3) &= 0.1061 \approx 1/16 \end{aligned}$$

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問題

- 遮断周波数 $\omega_c = 3\pi/4$, 3次FIR型のLPFを設計せよ

Quiz

- Design 3rd order FIR filter having cut-off frequency $\omega_c = 3\pi/4$