信号理論

- No.3 離散フーリエ変換とz変換 -

渡辺　裕

Signal Theory
- No.3 Discrete Fourier Transform and z-transform -

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離散フーリエ変換

- 離散信号のフーリエ変換および逆変換は

$$X(\omega)=\int_{-\infty}^{\infty}x(t)\exp(-j\omega t)dt$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\exp(j\omega t)d\omega$$

を離散化して

$$X(\omega)=\sum_{n=-\infty}^{\infty}x(n)\exp(-jn\omega)$$

$$x(n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\exp(jn\omega)d\omega$$

Discrete Fourier Transform

- Fourier Transform and Inverse Transform is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)d\omega$$

We can have discrete version as follows.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\exp(-jn\omega)$$

$$x(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(jn\omega)d\omega$$

離散フーリエ変換 (2)

- フーリエ変換は信号 $x(n)$ のパワーが有限であるとき意味を持つから，$n=0, 1,\ldots, N-1$ とし，ω も有限な L 個のサンプルで与えられるとき，　$L=N$ ならば，離散フーリエ変換対は次式のように定義される．

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk/N)$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \exp(j2\pi nk/N)$$

離散的な角周波数

$$\omega_k = \frac{2\pi k}{N} \quad (k = 0, 1, \cdots, N-1)$$

Discrete Fourier Transform (2)

- Fourier Transform has the meaning when the input signal power is limited; such as the input $x(n)$ is limited as $n=0, 1, \ldots, N-1$ and frequency ω is also limited as L sample. Fourier Transform Pair can be defined if $L = N$.

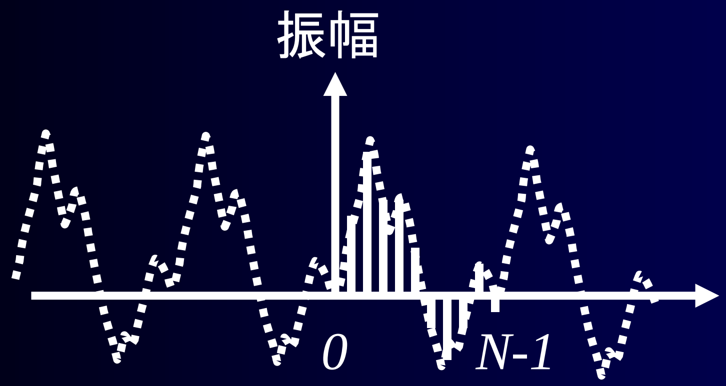
$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk/N)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp(j2\pi nk/N)$$

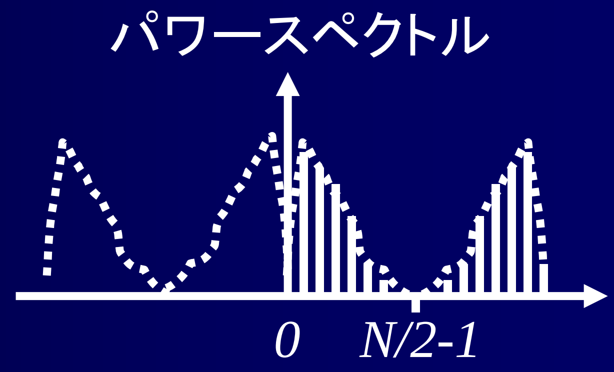
Discrete frequency $\omega_k = \dfrac{2\pi k}{N} \quad (k = 0, 1, \cdots, N-1)$

離散フーリエ変換の特徴

- 離散フーリエ変換, 逆変換をそれぞれ DFT (Discrete Fourier Transform), IDFT (Inverse DFT) と呼ぶ.
- 入力信号 $x(n)$ もフーリエ変換係数 $X(k)$ も, 区間 $[0, N\text{-}1]$ で離散的な値をとる. それ以外の範囲では, ともに周期 N で繰り返す波形となる.



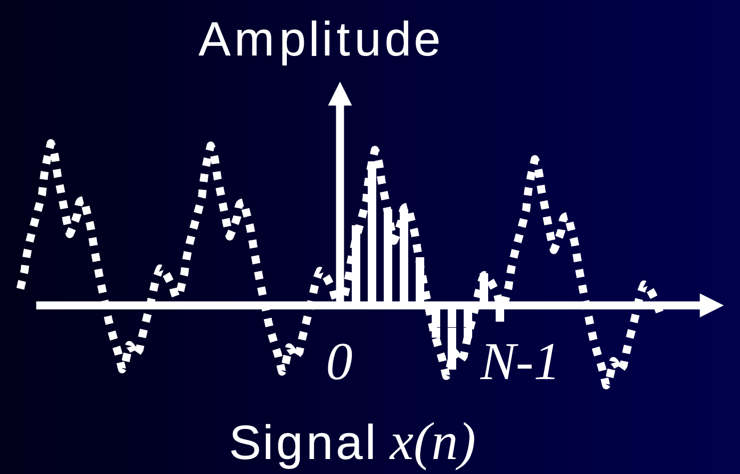
信号 $x(n)$

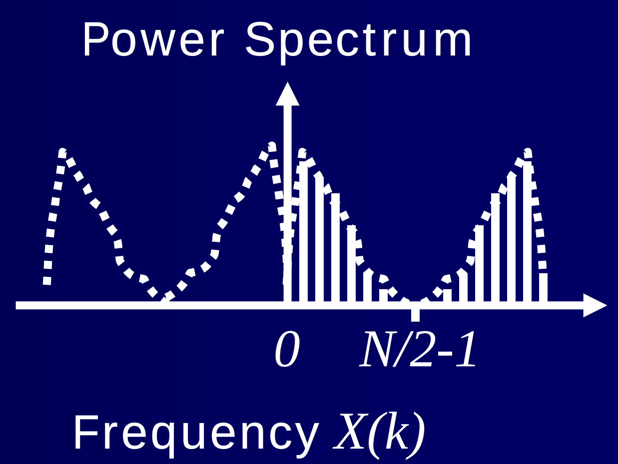


周波数領域 $X(k)$

Characteristic of Discrete Fourier Transform

- Transform Pair is called as DFT (Discrete Fourier Transform), IDFT (Inverse DFT).
- Both the input $x(n)$ and Fourier Transform Coefficient $X(k)$ have discrete values in the range of $[0, N\text{-}1]$. Outside of this range, waves repeated in the period N.





離散フーリエ変換の特徴 (2)

■ Nが偶数のとき，x(n)のDFTにおいて

$$X\left(\frac{N}{2}+m\right)=X^*\left(\frac{N}{2}-m\right) \qquad (m=0,1,\cdots,\frac{N}{2})$$

*:複素共役

が成り立つ．なぜなら

$$X\left(\frac{N}{2}+m\right)=\sum_{n=0}^{N-1}x(n)exp\left(-j2n\pi\left(\frac{N}{2}+m\right)/N\right)$$

$$=\sum_{n=0}^{N-1}x(n)exp\left(j2n\pi\left(\frac{N}{2}-m\right)/N\right)exp\left(-j2n\pi\right)$$

$$=\sum_{n=0}^{N-1}x(n)exp\left(j2n\pi\left(\frac{N}{2}-m\right)/N\right)$$

$$=X^*\left(\frac{N}{2}-m\right)$$

Characteristics of Discrete Fourier Transform (2)

- When N is even, at the DFT of $x(n)$, it holds

$$X\left(\frac{N}{2}+m\right)=X^*\left(\frac{N}{2}-m\right) \qquad (m=0,1,\cdots,\frac{N}{2})$$

*:Complex conjugate

since

$$\begin{aligned}
X\left(\frac{N}{2}+m\right) &= \sum_{n=0}^{N-1} x(n)exp\left(-j2n\pi\left(\frac{N}{2}+m\right)/N\right) \\
&= \sum_{n=0}^{N-1} x(n)exp\left(j2n\pi\left(\frac{N}{2}-m\right)/N\right)exp(-j2n\pi) \\
&= \sum_{n=0}^{N-1} x(n)exp\left(j2n\pi\left(\frac{N}{2}-m\right)/N\right) \\
&= X^*\left(\frac{N}{2}-m\right)
\end{aligned}$$

周波数解析

- 離散フーリエ変換は信号の周期性を仮定
- 信号の周波数解析に用いる場合には切り出し部分の両端で不連続となる
- 不連続を低減するために“窓関数”を用いる

- 波形$x(t)$に窓関数$w(t)$を掛け，$y(t)=w(t)x(t)$ となる

$$Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega-\tau)X(\tau)d\tau$$

$$Y(k) = \frac{1}{N}\sum_{l=0}^{N-1} W(k-l)X(l)$$

Frequency Analysis

- Discrete Fourier Transform assumes periodic signal
- Discontinuity occurs when a part of signal is cut out
- “Window function” is applied to reduce discontinuity

- Modified signal $y(t)=w(t)x(t)$ can be obtained from signal $x(t)$ by multiplying $w(t)$

$$Y(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}W(\omega-\tau)X(\tau)d\tau$$

$$Y(k)=\frac{1}{N}\sum_{l=0}^{N-1}W(k-l)X(l)$$

周波数解析 (2)

- 方形窓

$$w(nT)=\begin{cases}1 & |n|\le \dfrac{N-1}{2}\\ 0 & otherwise\end{cases}$$

- Hamming窓

$$w(nT)=\begin{cases}0.54+0.46\cos\left(\dfrac{2\pi n}{N-1}\right) & |n|\le \dfrac{N-1}{2}\\ 0 & otherwise\end{cases}$$

Frequency Analysis (2)

- Rectangular Window

$$w(nT) = \begin{cases} 1 & |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases}$$

- Hamming Window

$$w(nT) = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{2\pi n}{N-1}\right) & |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases}$$

周波数解析 (3)

- Hann(ing)窓

$$w(nT) = \begin{cases} 0.5\left[1 + \cos\left(\dfrac{2\pi n}{N-1}\right)\right] & |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases}$$

- Blackman窓

$$w(nT) = \begin{cases} 0.42 + 0.5\cos\left(\dfrac{2\pi n}{N-1}\right) + 0.08\cos\left(\dfrac{4\pi n}{N-1}\right) & |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases}$$

Frequency Analysis (3)

- Hann(ing) Window

$$w(nT) = \begin{cases} 0.5\left[1+\cos\left(\dfrac{2\pi n}{N-1}\right)\right] & |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases}$$

- Blackman Window

$$w(nT) = \begin{cases} 0.42+0.5\cos\left(\dfrac{2\pi n}{N-1}\right)+0.08\cos\left(\dfrac{4\pi n}{N-1}\right) & |n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases}$$

高速フーリエ変換

- 基数2のFFTアルゴリズム(N＝2^m)

$$y(n) = x(2n) \qquad n = 0,1,\cdots,\frac{N}{2}-1$$
$$z(n) = x(2n+1)$$

- $W_N = exp(-j2\pi/N)$とすると$x(n)$のフーリエ変換の結果

$$\begin{aligned} X(k) &= \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{(2n+1)k} \\ &= \sum_{n=0}^{N/2-1} y(n) W_{N/2}^{nk} + W_{N/2}^{k} \sum_{n=0}^{N/2-1} z(n) W_{N/2}^{nk} \\ &\equiv Y(k) + W_N^k Z(k) \end{aligned}$$

Fast Fourier Transform

- FFT Algorithm with Radix 2 ($N=2^m$)

$$y(n) = x(2n) \qquad n = 0,1,\cdots,\frac{N}{2}-1$$
$$z(n) = x(2n+1)$$

- Result of Fourier Transform *of x(n)* when $W_N = exp(-j2\pi/N)$

$$X(k) = \sum_{n=0}^{N/2-1} x(2n)W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1)W_N^{(2n+1)k}$$
$$= \sum_{n=0}^{N/2-1} y(n)W_{N/2}^{nk} + W_{N/2}^{k} \sum_{n=0}^{N/2-1} z(n)W_{N/2}^{nk}$$
$$\equiv Y(k) + W_N^k Z(k)$$

高速フーリエ変換 (2)

- Y(k), Z(k)はN/2点のDFTであり、対称性

$$W_N^k = -W_N^{k-N/2}$$

より以下を得る

$$X(k) = \begin{cases} Y(k) + W_N^k Z(k) & 0 \le k \le \dfrac{N}{2} - 1 \\ Y\left(k - \dfrac{N}{2}\right) - W_N^{k-N/2} Z\left(k - \dfrac{N}{2}\right) & \dfrac{N}{2} \le k \le N - 1 \end{cases}$$

Fast Fourier Transform (2)

- Y(k), Z(k) are N/2 point DFT, this symetricity holds

$$W_N^k = -W_N^{k-N/2}$$

Thus,

$$X(k) = \begin{cases} Y(k) + W_N^k Z(k) & 0 \le k \le \dfrac{N}{2} - 1 \\ Y\left(k - \dfrac{N}{2}\right) - W_N^{k-N/2} Z\left(k - \dfrac{N}{2}\right) & \dfrac{N}{2} \le k \le N - 1 \end{cases}$$

高速フーリエ変換 (3)

- 例: N＝8

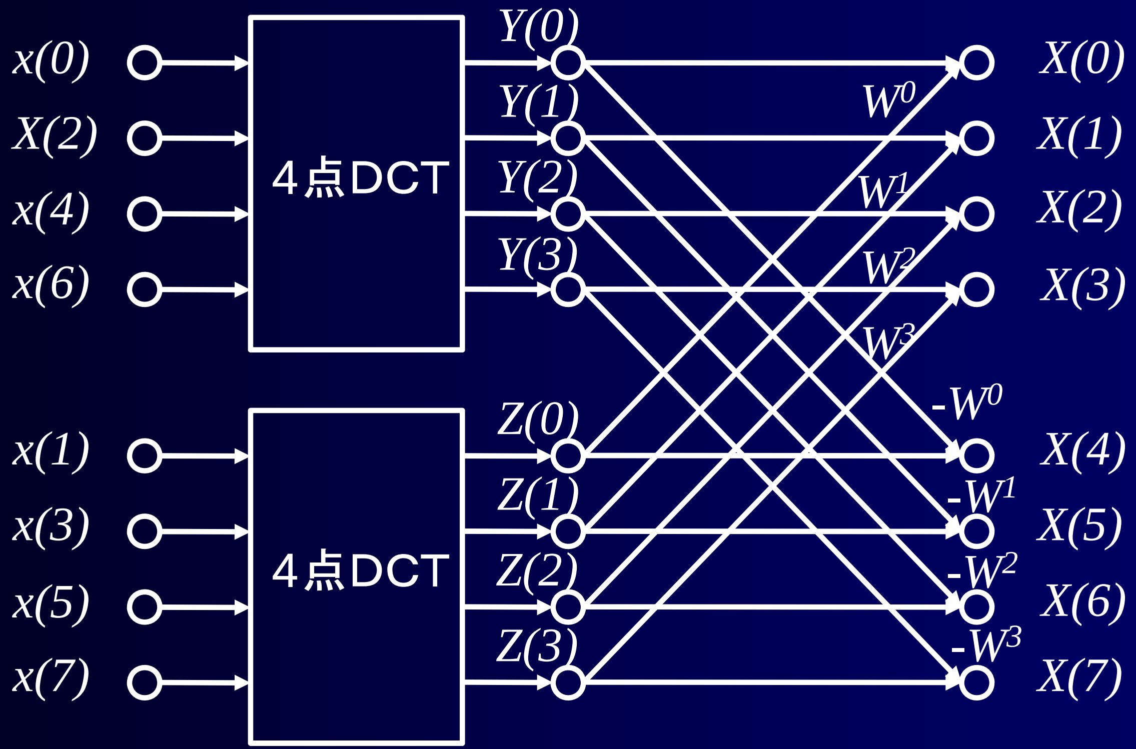
$$
\begin{aligned}
X(0) &= Y(0) + W^0 Z(0) & X(4) &= Y(0) - W^0 Z(0) \\
X(1) &= Y(1) + W^1 Z(1) & X(5) &= Y(1) - W^1 Z(1) \\
X(2) &= Y(2) + W^2 Z(2) & X(6) &= Y(2) - W^2 Z(2) \\
X(3) &= Y(3) + W^3 Z(3) & X(7) &= Y(3) - W^3 Z(3)
\end{aligned}
$$

Fast Fourier Transform (3)

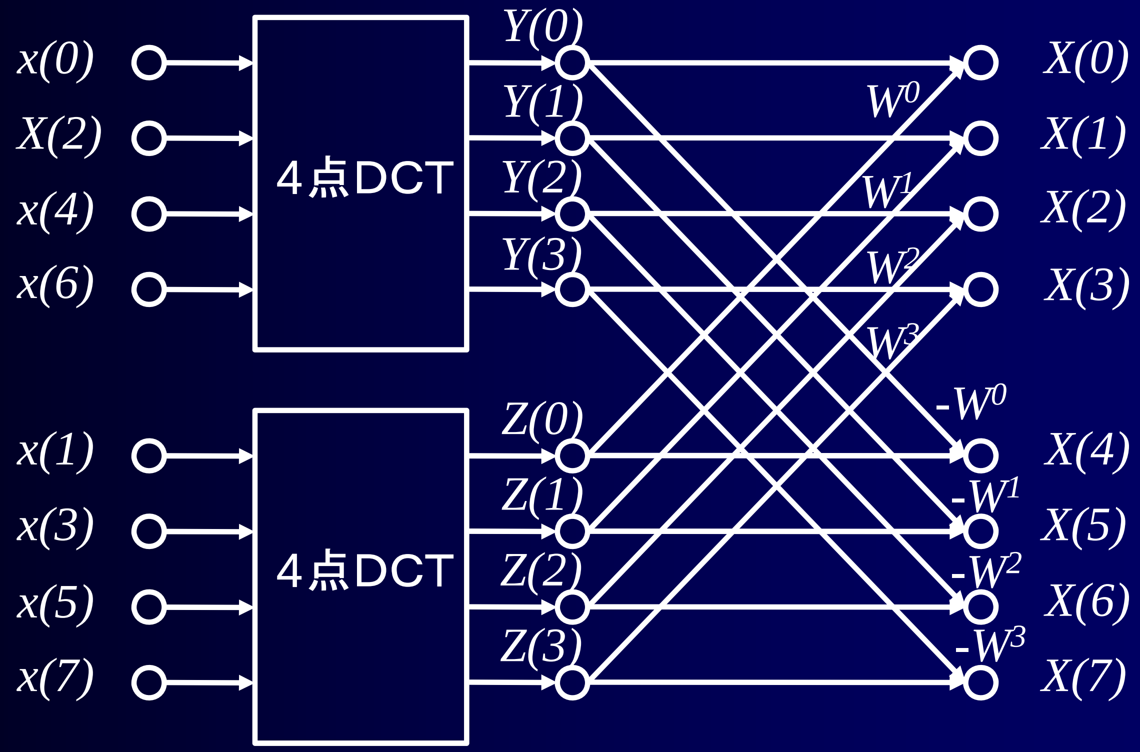
- Example: N=8

$$X(0) = Y(0) + W^0 Z(0) \qquad X(4) = Y(0) - W^0 Z(0)$$
$$X(1) = Y(1) + W^1 Z(1) \qquad X(5) = Y(1) - W^1 Z(1)$$
$$X(2) = Y(2) + W^2 Z(2) \qquad X(6) = Y(2) - W^2 Z(2)$$
$$X(3) = Y(3) + W^3 Z(3) \qquad X(7) = Y(3) - W^3 Z(3)$$

バタフライ演算


x(0)
X(2)
x(4)
x(6)
4点DCT
x(1)
x(3)
x(5)
x(7)
4点DCT
Y(0)
Y(1)
Y(2)
Y(3)
Z(0)
Z(1)
Z(2)
Z(3)
W0
W1
W2
W3
-W0
-W1
-W2
-W3
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)

Butterfly Operation



z変換

- 入力信号 $x(n)$ に対して，サンプリングに関する単位遅延 $z=exp(j\omega/2f_m)$ を規定すると，$x(n)$ のz変換および逆変換は次式で定義される．

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$x(n)=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

C:複素平面における単位円上の周回積分

Z-Transform

- To the input $x(n)$, by defining the unit delay regarding sampling as $z=exp(j\omega/2f_m)$, z transform and the Inverse transform of $x(n)$ can be given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

C: Circular integration on the unit circle
on the complex domain

z変換 (2)

- 離散時間信号 *x(n)* に対する定常線形システム(フィルター)は，入力信号 *x(t)* とフィルタのインパルス応答 *h(n)* の畳み込みで表される

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

- インパルス応答 *h(n)* において *n* が有限であれば，FIR (Finite Impulse Response) フィルタと呼ばれ，無限に継続すれば，IIR (Infinite Impulse Response) フィルタと呼ばれる.

$$Y(z) = H(z)X(z)$$

Z-transform (2)

- Linear invariant system (filter) to the discrete signal *x(n)* can be represented by convolution of the input *x(t)* and filter's impulse response *h(n)*.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

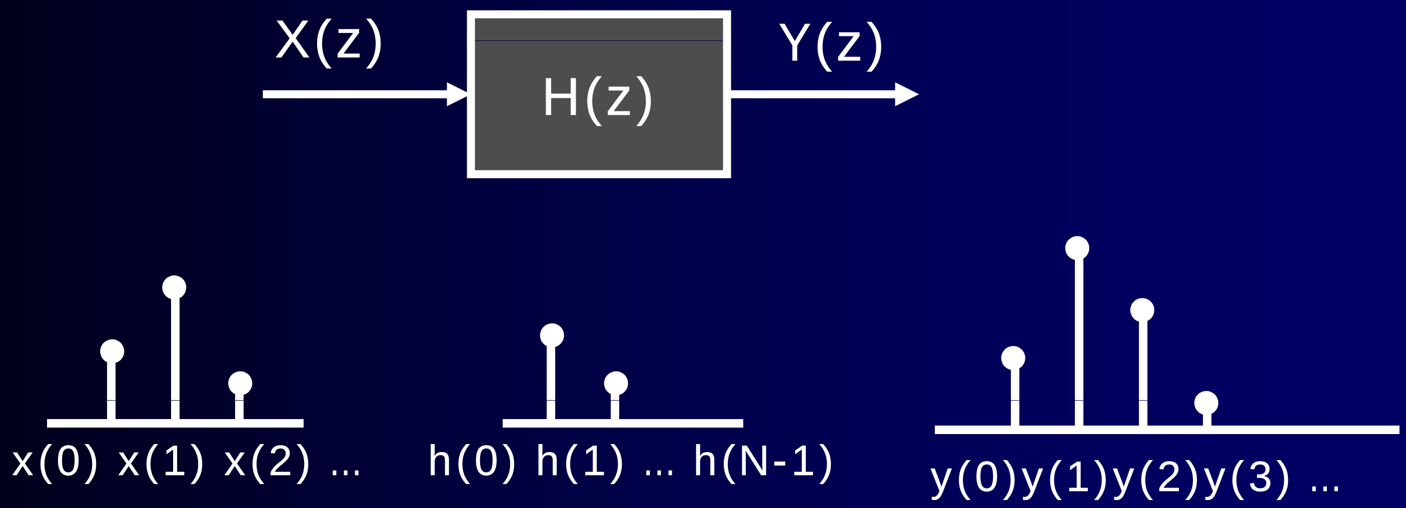
- The impulse response *h(n)* is called FIR (Finite Impulse Response) filter when *n* is limited, and is called IIR (Infinite Impulse Response) filter if *n* is not limited.

$$Y(z) = H(z)X(z)$$

z変換 (3)

- 定常線形システムのz変換による表記

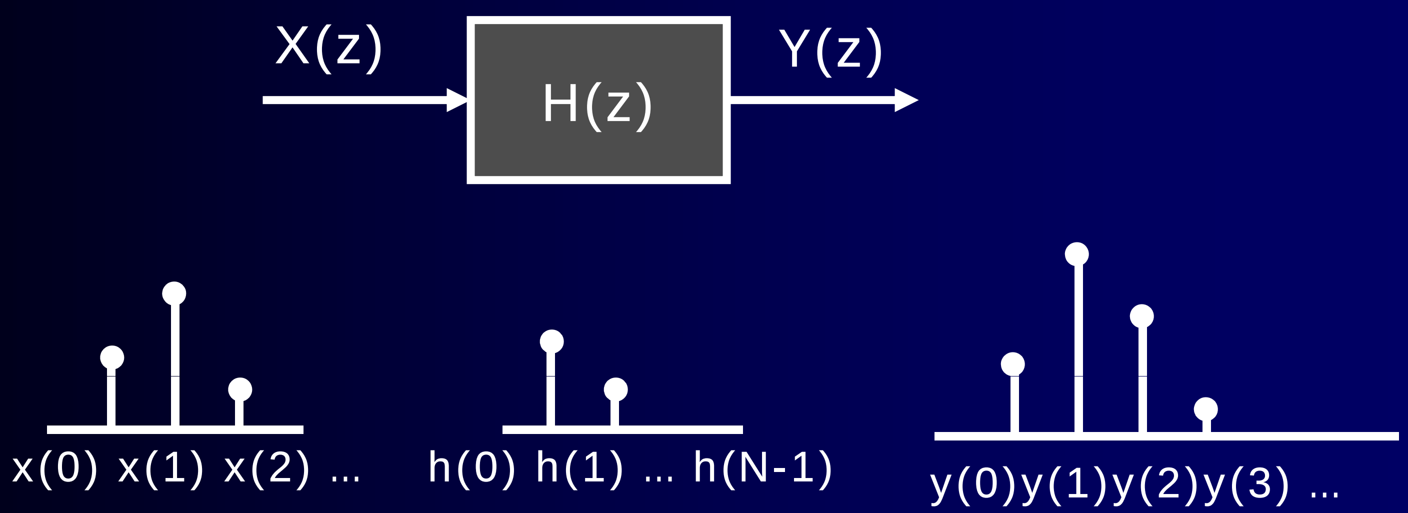
$$Y(z) = H(z)X(z)$$



Z-transform (3)

- Representation of linear invariant system by z transform.

$$Y(z) = H(z)X(z)$$



係数の対応

- 入力信号

$$X(z)=a_0+a_1z^{-1}+a_2z^{-2}+a_3z^{-3}$$

- インパルス応答

$$H(z)=h_0+h_1z^{-1}+h_2z^{-2}$$

- 出力信号

$$\begin{aligned}Y(z)&=H(z)X(z)\\&=(h_0+h_1z^{-1}+h_2z^{-2})(a_0+a_1z^{-1}+a_2z^{-2}+a_3z^{-3})\\&=h_0a_0+(h_0a_1+h_1a_0)z^{-1}+(h_0a_2+h_1a_1+h_2a_0)z^{-2}\\&\quad+(h_0a_3+h_1a_2+h_2a_1)z^{-3}+(h_1a_3+h_2a_2)z^{-4}+h_2a_3z^{-5}\end{aligned}$$

Coefficients

- Input $X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$
- Impulse Response $H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2}$
- Output

$$
\begin{aligned}
Y(z) &= H(z)X(z) \\
&= (h_0 + h_1 z^{-1} + h_2 z^{-2})(a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}) \\
&= h_0 a_0 + (h_0 a_1 + h_1 a_0)z^{-1} + (h_0 a_2 + h_1 a_1 + h_2 a_0)z^{-2} \\
&\quad + (h_0 a_3 + h_1 a_2 + h_2 a_1)z^{-3} + (h_1 a_3 + h_2 a_2)z^{-4} + h_2 a_3 z^{-5}
\end{aligned}
$$

係数の対応 (2)

- 係数の関係

$$y_0 = h_0 a_0$$
$$y_1 = h_0 a_1 + h_1 a_0$$
$$y_2 = h_0 a_2 + h_1 a_1 + h_2 a_0$$
$$y_3 = h_0 a_3 + h_1 a_2 + h_2 a_1$$
$$y_4 = h_1 a_3 + h_2 a_2$$
$$y_5 = h_2 a_3$$

Coefficients (2)

- Relation of Coefficients

$$
\begin{aligned}
y_0 &= h_0 a_0 \\
y_1 &= h_0 a_1 + h_1 a_0 \\
y_2 &= h_0 a_2 + h_1 a_1 + h_2 a_0 \\
y_3 &= h_0 a_3 + h_1 a_2 + h_2 a_1 \\
y_4 &= h_1 a_3 + h_2 a_2 \\
y_5 &= h_2 a_3
\end{aligned}
$$

線形差分方程式

- 離散時間システム

$$\sum_{i=0}^{N} a_i y(n-i) = \sum_{j=0}^{M} b_j x(n-j)$$

- Z変換を適用

$$Y(z)\sum_{i=0}^{N} a_i z^{-i} = X(z)\sum_{j=0}^{M} b_j z^{-j}$$

- 伝達関数

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{j=0}^{M} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}}$$

Linear Differential Equation

- Discrete time system

$$\sum_{i=0}^{N} a_i y(n-i) = \sum_{j=0}^{M} b_j x(n-j)$$

- Application of z-transform

$$Y(z)\sum_{i=0}^{N} a_i z^{-i} = X(z)\sum_{j=0}^{M} b_j z^{-j}$$

- Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{j=0}^{M} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}}$$

z変換の特徴

- インパルス信号 $x(n)=\delta(n)$ のz変換

$$X(z)=1$$

- ステップ信号 $x(n)=\begin{cases} 1 & n\geq 0 \\ 0 & n<0 \end{cases}$ のz変換

$$X(z)=\frac{1}{1-z^{-1}}$$

Characteristics of z-transform

- z transform of the impulse signal $x(n)=\delta(n)$

$$X(z)=1$$

- z transform of the step signal

$$x(n)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$$

$$X(z)=\frac{1}{1-z^{-1}}$$

Z変換の特徴 (2)

- 信号 $x(n)=\begin{cases} a^n & n \ge 0 \\ 0 & n < 0 \end{cases}$ のz変換

$$X(z)=\frac{1}{1-az^{-1}}$$

- $z(n)=ax(n)+by(n)$ のz変換は

$$Z(z)=aX(z)+bY(z)$$

Characteristics of z-transform (2)

- z transform of the signal $x(n)=\begin{cases} a^n & n\geq 0 \\ 0 & n<0 \end{cases}$

$$X(z)=\frac{1}{1-az^{-1}}$$

- z transform of the signal $z(n)=ax(n)+by(n)$

$$Z(z)=aX(z)+bY(z)$$