信号理論

- No.1 標本化定理 -

渡辺　裕

Signal Theory

- No.1 Sampling Theorem -

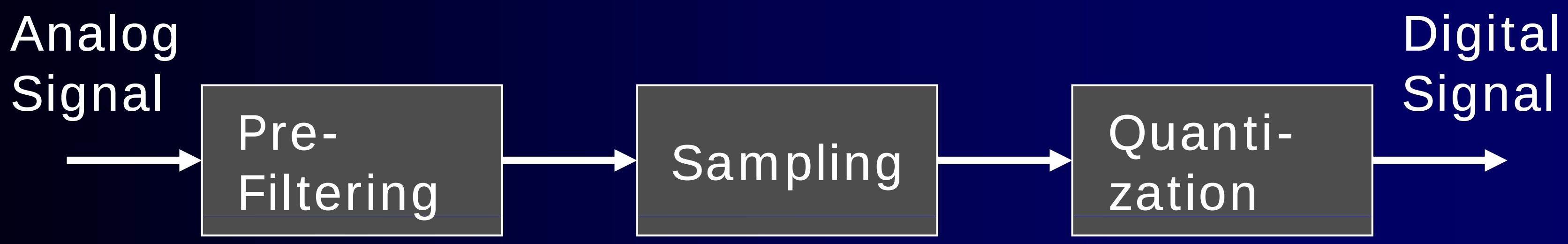
Hiroshi Watanabe

A/D 変換

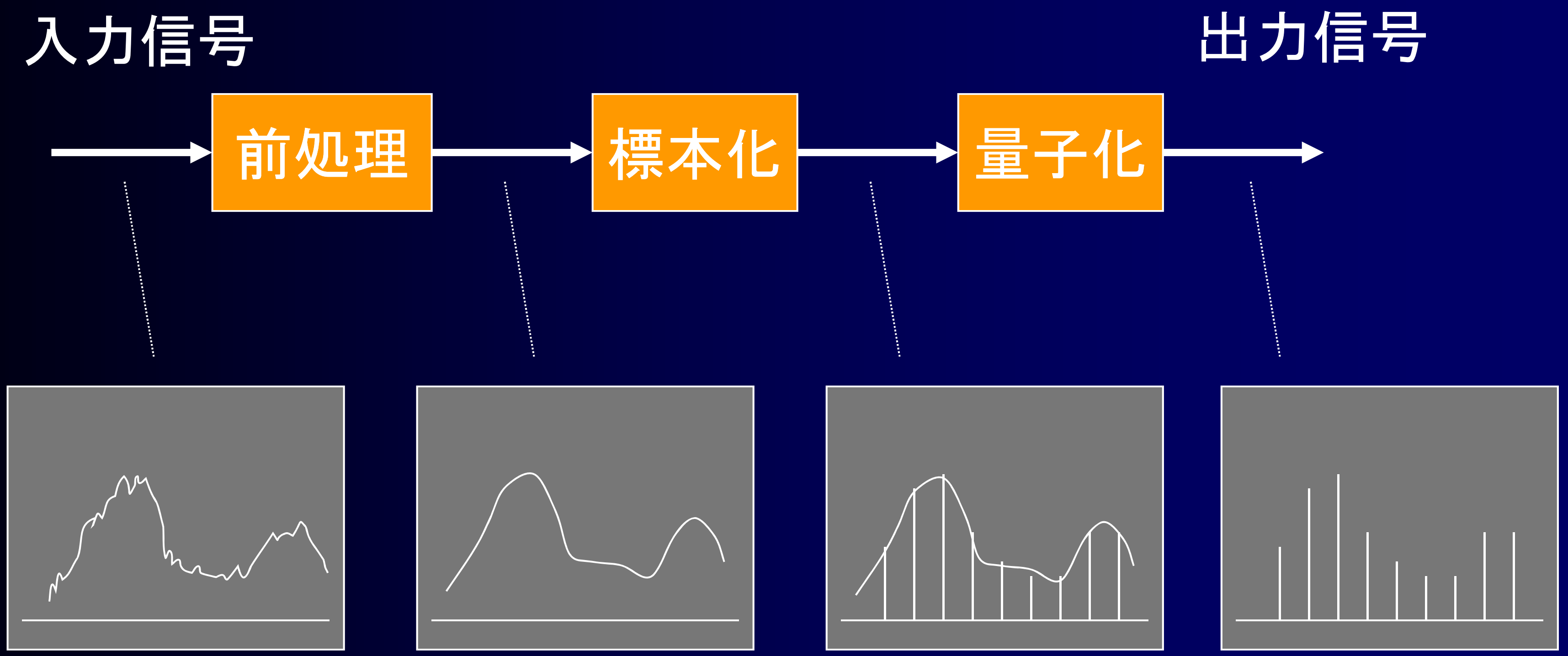
- アナログ信号からディジタル信号へ
- プレフィルタ処理：高周波成分の除去
- 標本化：時間軸における離散化
- 量子化：レベル値の有限n進数への変換

A/D Conversion

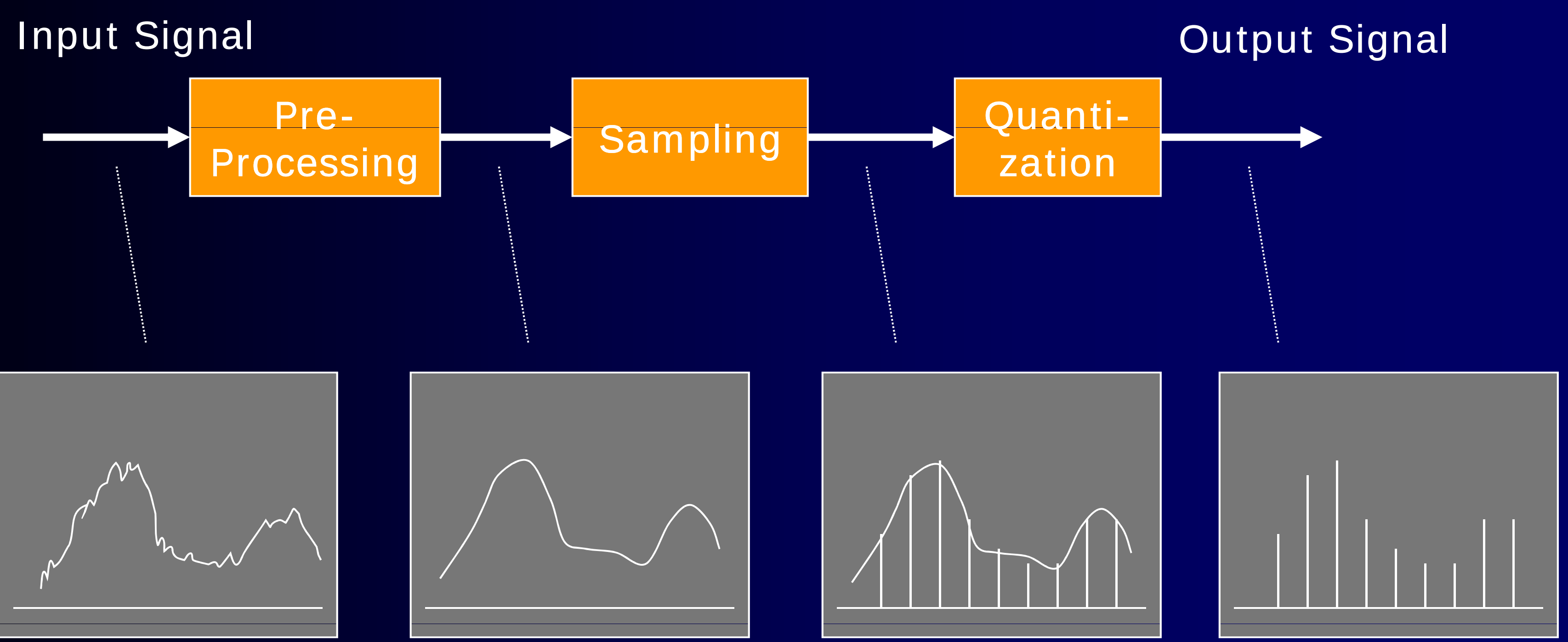
- Analog Signal to Digital Signal
- Pre-filtering: Cut High Frequency Component
- Sampling: Discrete in Time Domain
- Quantization: Convert Value to Finite Digits



処理ダイヤグラム

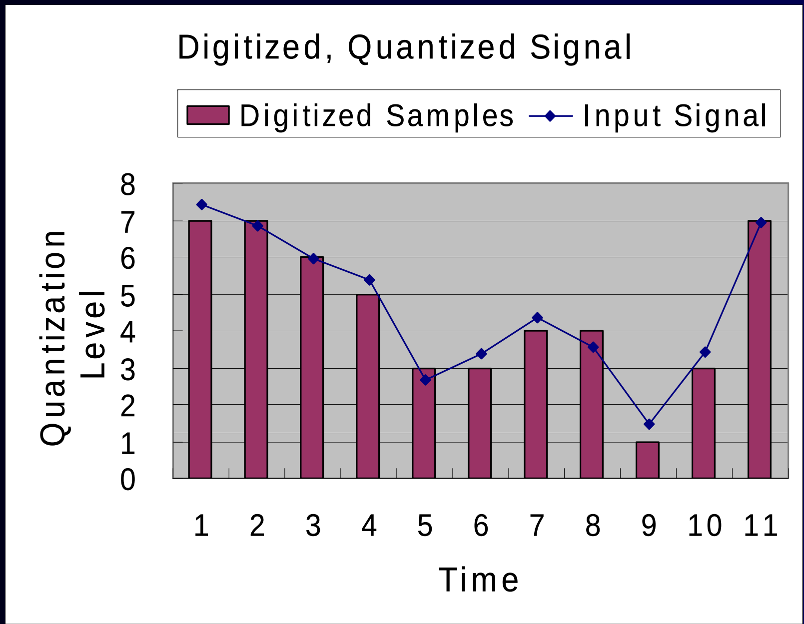


Processing Diagram

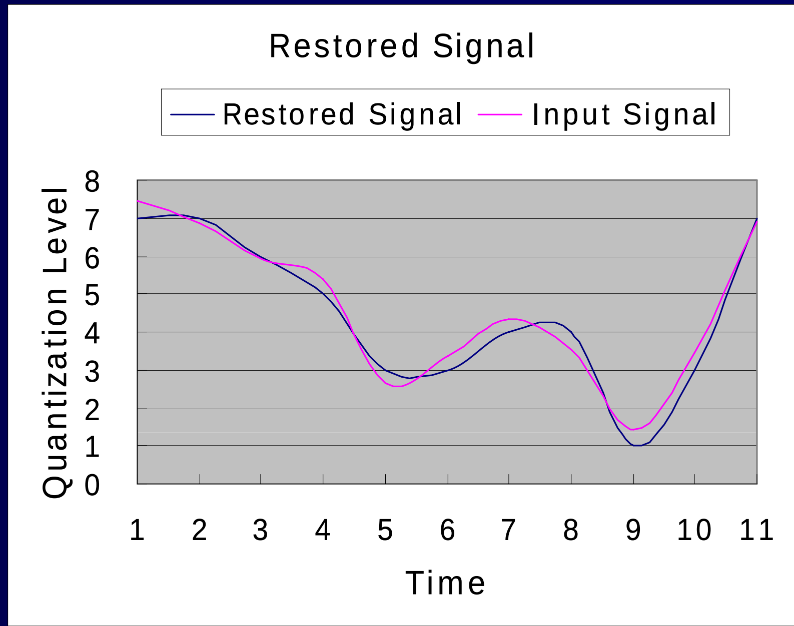


PCM

- PAMデータへのユニーク符号の割り当て



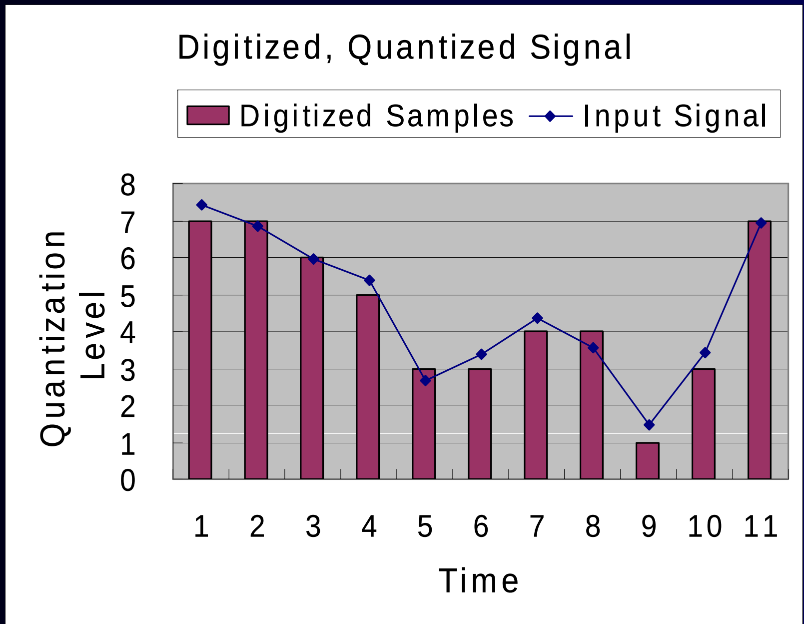
離散化・量子化信号

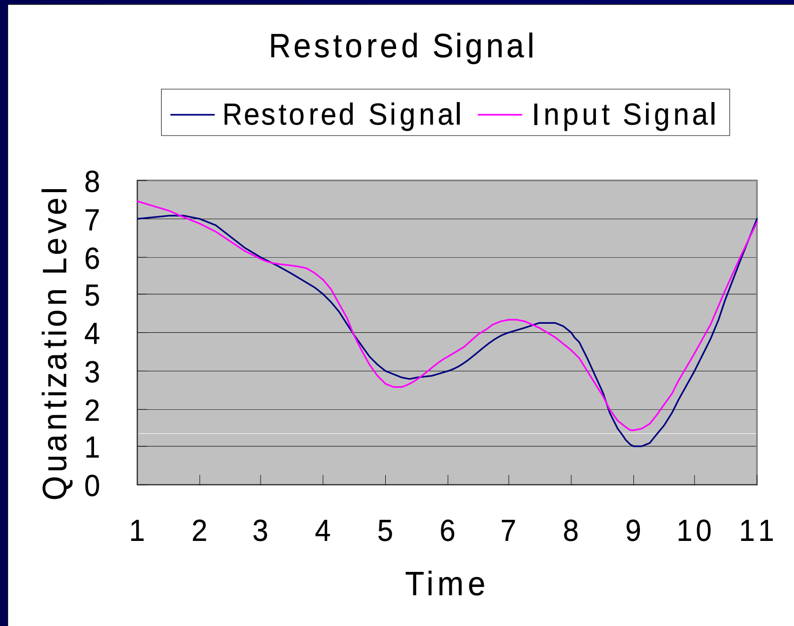


再生信号

PCM

- Unique Code Assignment to PAM Data





Digitized and quantized signal

Restored signal

2進数と10進数

- 2進数表現

$$b_n b_{n-1} \cdots b_1 b_0 \,.\, b_{-1} b_{-2} \cdots b_{-m} \Big|_2 = \left. \begin{array}{l} b_n 2^n + b_{n-1} 2^{n-1} + \cdots + b_1 2^1 + b_0 2^0 \\ + b_{-1} 2^{-1} + \cdots + b_{-m} 2^{-m} \end{array} \right|_{10}$$

$$= \sum_{i=-m}^{n} b_i 2^i \Big|_{10} \quad , \quad b_i = \begin{cases} 0 \\ 1 \end{cases}$$

- 例

$$10110 = 2^4 + 2^2 + 2^1 = 16 + 4 + 2 = 22$$

Binary and Decimal Numbers

- Binary Representation

$$b_n b_{n-1} \cdots b_1 b_0 \,.\, b_{-1} b_{-2} \cdots b_{-m} \Big|_2 = \left. \begin{array}{l} b_n 2^n + b_{n-1} 2^{n-1} + \cdots + b_1 2^1 + b_0 2^0 \\ + b_{-1} 2^{-1} + \cdots + b_{-m} 2^{-m} \end{array} \right|_{10}$$

$$= \sum_{i=-m}^{n} b_i 2^i \Big|_{10} \quad , \quad b_i = \begin{cases} 0 \\ 1 \end{cases}$$

- Example

$$10110 = 2^4 + 2^2 + 2^1 = 16 + 4 + 2 = 22$$

2進符号

- レベル数とビット数

Level数	Bit	Code
2	1	0, 1
4	2	00, 01, 10, 11
8	3	000, 001, 010, 011, 100, 101, 110, 111
...		
2^n	n	00...0, ...

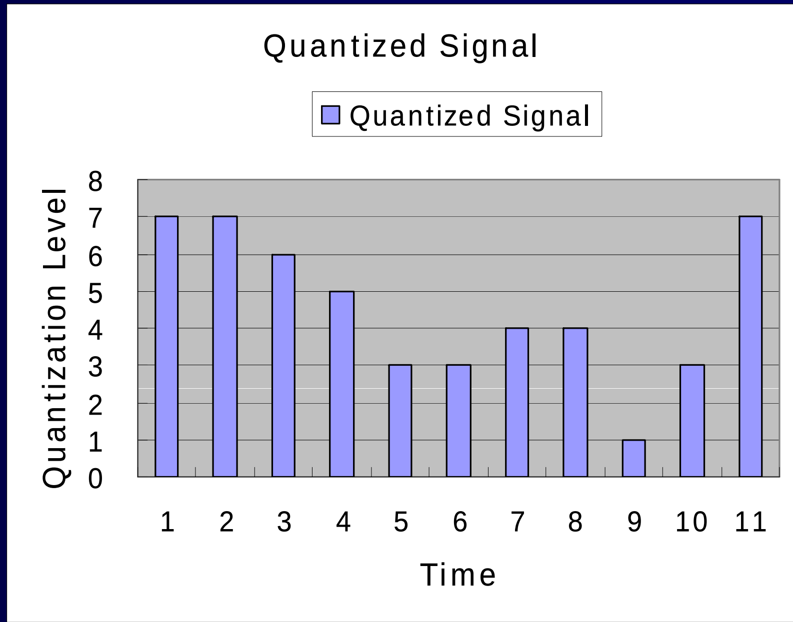
Binary Code

- Level and Bit Numbers

Level#s	Bit	Code
2	1	0, 1
4	2	00, 01, 10, 11
8	3	000, 001, 010, 011, 100, 101, 110, 111
...		
2^n	n	00...0, ...

PCM符号の割り当て

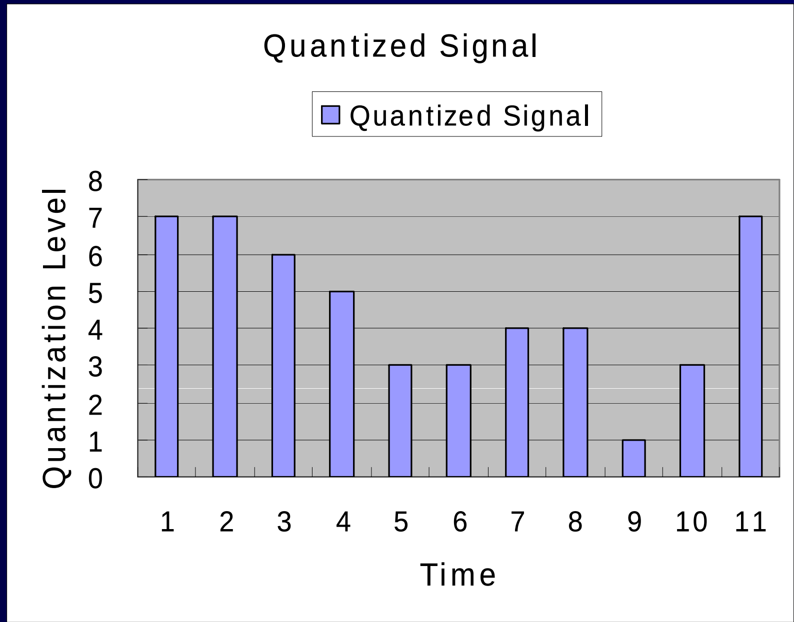
- Level　Codeword
  - 0　000
  - 1　001
  - 2　010
  - 3　011
  - 4　100
  - 5　101
  - 6　110
  - 7　111



時刻:	1	2	3	4	5	6	7	8	9	10	11	
レベル:	7	7	6	5	3	3	4	4	1	3	7	
符号:	111	111	110	101	011	011	100	100	001	011	111	/ 33 bit

Assignment of PCM Code

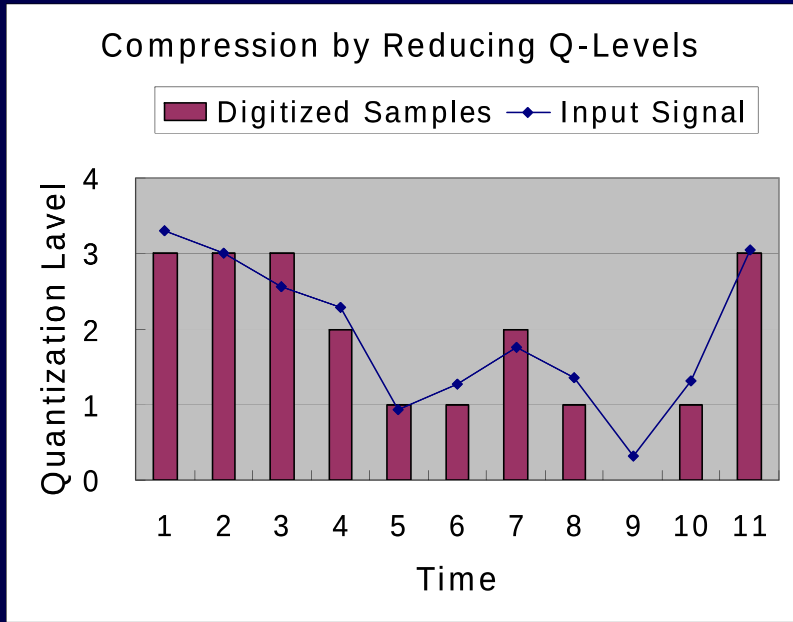
- Level   Codeword
  - 0   000
  - 1   001
  - 2   010
  - 3   011
  - 4   100
  - 5   101
  - 6   110
  - 7   111



Time:	1	2	3	4	5	6	7	8	9	10	11
Level:	7	7	6	5	3	3	4	4	1	3	7
Code:	111	111	110	101	011	011	100	100	001	011	111 / 33 bit

基本的なデータ圧縮 – 量子化レベル数の削減

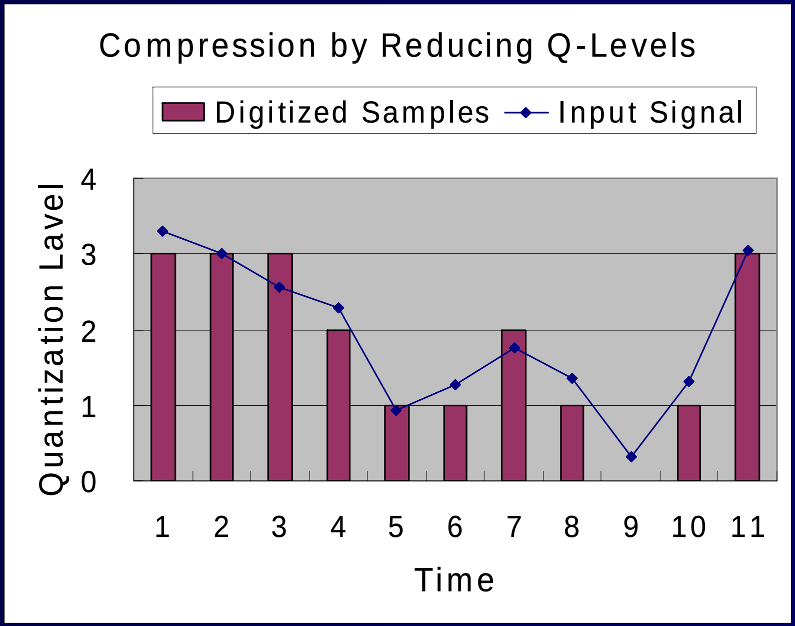
Level	Codeword
0	00
1	01
2	10
3	11



時刻:	1	2	3	4	5	6	7	8	9	10	11
レベル:	3	3	3	2	1	1	2	1	0	1	3
符号:	11	11	11	10	01	01	10	10	00	10	11 / 22bit

Basic Data Compression –
Reducing Number of Quantization Levels

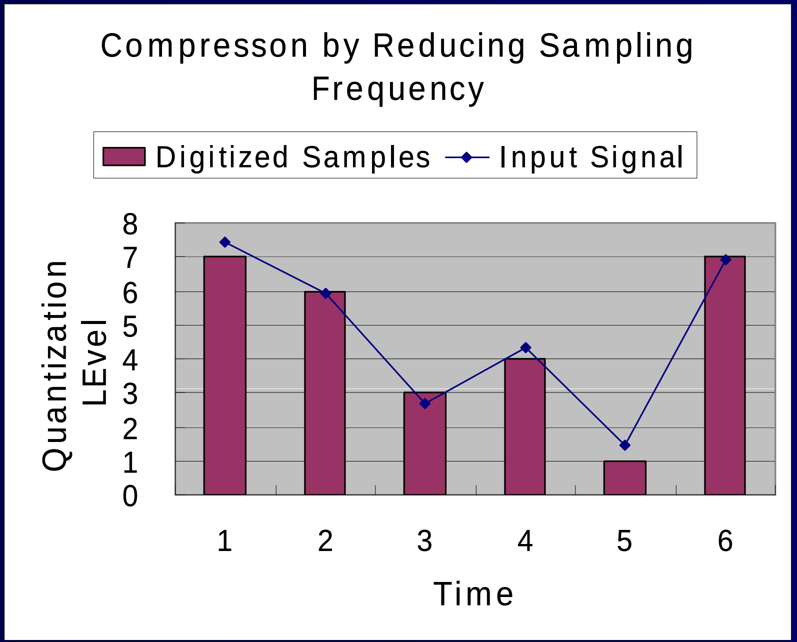
Level	Codeword
0	00
1	01
2	10
3	11



Time:	1	2	3	4	5	6	7	8	9	10	11
Level:	3	3	3	2	1	1	2	1	0	1	3
Code:	11	11	11	10	01	01	10	10	00	10	11 / 22bit

基本的なデータ圧縮 – 標本点数の削減

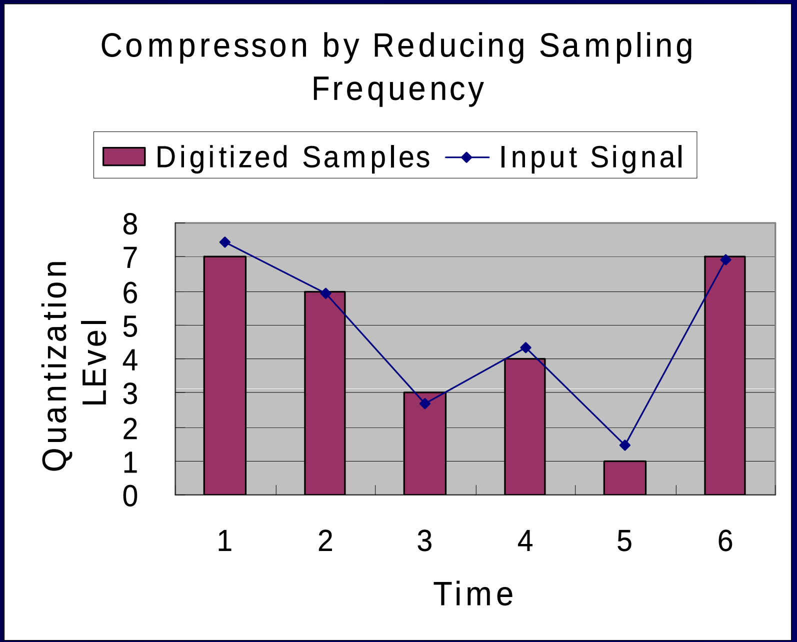
Level	Codeword
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111



時刻: 1 2 3 4 5 6
レベル: 7 6 3 4 1 7
符号: 111 110 011 100 001 111 / 18 bit

Basic Data Compression –
Reducing Number of Samples

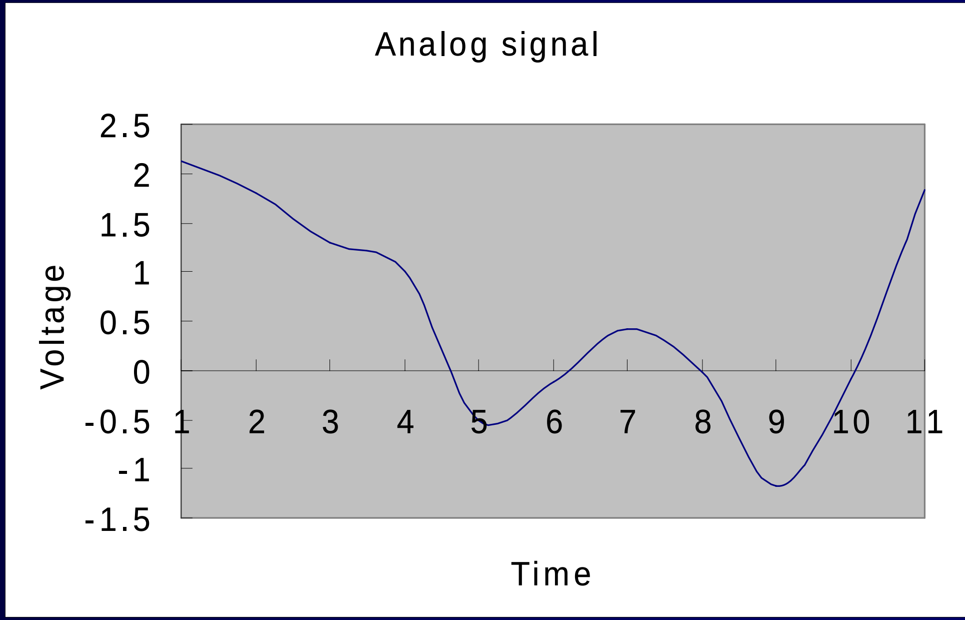
Level	Codeword
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111



Time:	1	2	3	4	5	6
Level:	7	6	3	4	1	7
Code:	111	110	011	100	001	111 / 18 bit

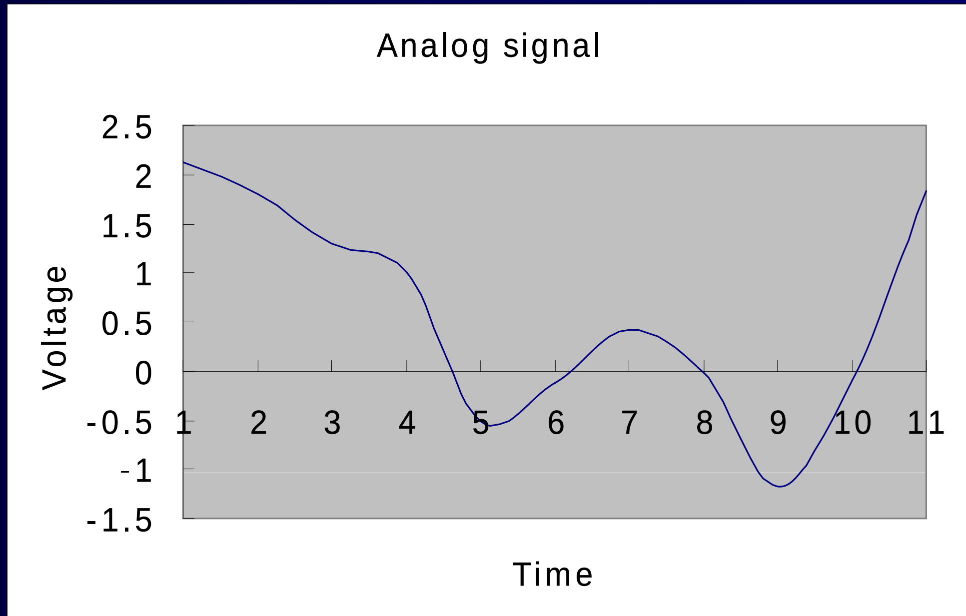
定量的解析

時刻	アナログ信号値
0	2.12
1	1.80
2	1.30
3	1.00
4	-0.50
5	-0.12
6	0.42
7	-0.02
8	-1.17
9	-0.08
10	1.84



Quantitative Analysis

Time	Analogue signal Value
0	2.12
1	1.80
2	1.30
3	1.00
4	-0.50
5	-0.12
6	0.42
7	-0.02
8	-1.17
9	-0.08
10	1.84



定量的解析 (2)

- 量子化ステップ ***d*** の算出

$$d = {}^{2V}\!/_{L}$$

V: 入力電圧の最大値
L: 量子化レベル数

$$L = 2^n$$

n : 符号のビット数

- 例

V=2.2, n=4 のとき L=16 であるから

$$d = 0.275$$

Quantitative Analysis (2)

- Calculation of Quantization step d

$$d = {}^{2V}\!/_{L}$$

V: Maximum value of input
L: number of quantization levels

$$L = 2^n$$

n : number of bits for code

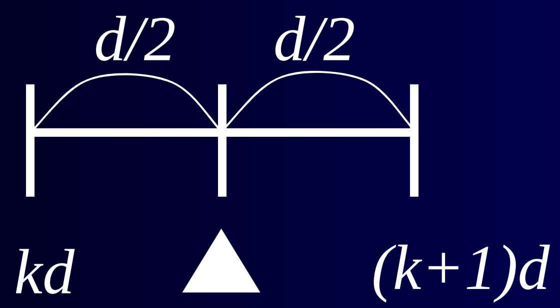
- Example

When V=2.2, n=4, L *equals to* 16, thus,

$$d = 0.275$$

定量的解析 (3)

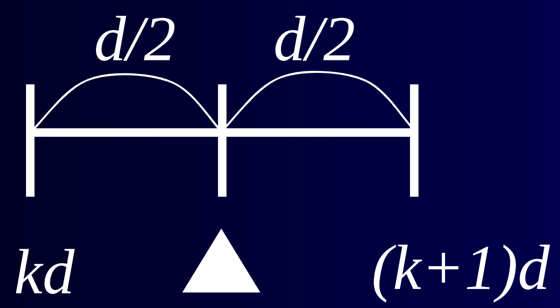
- 量子化誤差電力: σ_q^2



$$\sigma_q^2 = \frac{1}{d}\int_{-d/2}^{d/2} x^2 dx = \frac{1}{d}\left[\frac{1}{3}x^3\right]_{-d/2}^{d/2} = d^2\big/12$$

Quantitative Analysis (3)

- Quantization Error Power: σ_q^2



$$\sigma_q^2 = \frac{1}{d}\int_{-d/2}^{d/2} x^2 dx = \frac{1}{d}\left[\frac{1}{3}x^3\right]_{-d/2}^{d/2} = d^2 \big/ 12$$

定量的解析 (4)

- 量子化によるSNR(雑音対信号比)

$$SNR(dB) = 20\log_{10}\frac{\sigma_x}{\sigma_q} = 10\log_{10}\frac{\sigma_x^2}{\sigma_q^2}$$

- 入力信号の分散: σ_x^2 (画像処理では最大値)

$$\sigma_x^2 = 4V^2$$

Quantitative Analysis (4)

- SNR (Signal to Noise Ratio) caused by Quantization

$$SNR(dB) = 20\log_{10}\frac{\sigma_x}{\sigma_q} = 10\log_{10}\frac{\sigma_x^2}{\sigma_q^2}$$

- Variance of Input signal: σ_x^2 (maximum value for image processing)

$$\sigma_x^2 = 4V^2$$

定量的解析 (5)

- *n* ビット量子化時のSNRの算出

$$
\begin{aligned}
SNR(dB) &= 10\log_{10}\frac{4V^2}{d^2/12} \\
&= 10\log_{10}\frac{4V^2}{4V^2/12L^2} \\
&= 10\log_{10}12L^2 \\
&= 10(\log_{10}3 + 2\log_{10}2) + 20\log_{10}2^n \\
&= 10.79 + 6.02n
\end{aligned}
$$

$$
\log_{10}2 \cong 0.301
$$
$$
\log_{10}3 \cong 0.477
$$

1bit 6dBの法則

Quantitative Analysis (5)

- Calculation of SNR at n bit quantization

$$
\begin{aligned}
SNR(dB) &= 10\log_{10}\frac{4V^2}{d^2/12} \\
&= 10\log_{10}\frac{4V^2}{4V^2/12L^2} \\
&= 10\log_{10}12L^2 \\
&= 10(\log_{10}3 + 2\log_{10}2) + 20\log_{10}2^n \\
&= 10.79 + 6.02n
\end{aligned}
$$

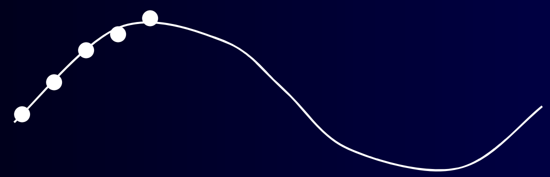
$$
\begin{aligned}
\log_{10}2 &\cong 0.301 \\
\log_{10}3 &\cong 0.477
\end{aligned}
$$

Rule: *1bit 6dB*

標本化

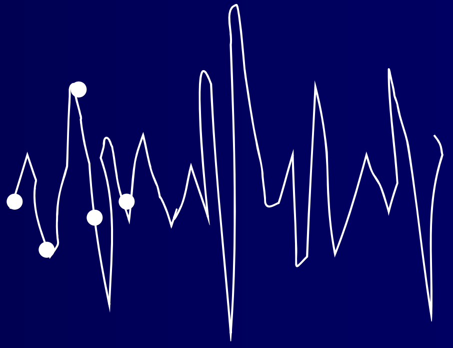
- 連続関数を一定時間間隔で標本化して得られる系列から元の連続関数を復元することができるか？

ゆっくり変化する入力



粗い標本でも復元できそう

急激に変化する入力



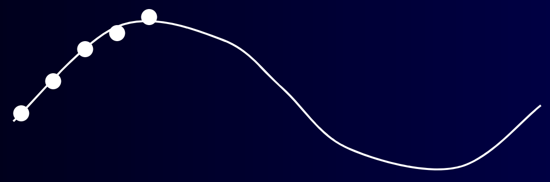
細かな標本化が必要そう

1928 Harry Nyquist 予測, 1949 Claude Shannon, 染谷勲 証明

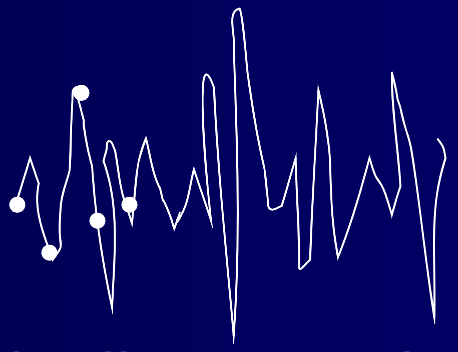
Sampling

- Can we recover original analog signal from sampled digital data?

Slowly Changing Inputs



Rapidly Changing Inputs



Looks like recoverable even with coarse sampling

Looks like requesting fine sampling

1928 Harry Nyquist predicted, 1949 Claude Shannon & Isao Someya proved

サンプリング定理 (1)

- サンプリング間隔と入力信号の最高周波数には重要な関係がありそう

- シャノンのサンプリング定理(ナイキストの定理):
  - 時間関数 $x(t)$ の周波数成分が f_m(Hz) 以下に制限されているとき、サンプリング間隔を $1/(2f_m)$ 以下に制限すれば、得られたサンプル系列からもとの連続関数を完全に復元できる。

- ナイキスト間隔: $T=1/(2f_m)$, ナイキスト周波数: $2f_m$

Sampling Theorem (1)

- There is a significant relation between a sampling period and the maximum frequency of an input signal

- Shannon's Sampling Theorem (Nyquist Theorem):
  - "If the frequency component of an input signal $x(t)$ is less than f_m (Hz), we can recover the original continuous signal from sampled data by setting sampling interval less than $1/(2f_m)$"

- Nyquist Interval: $T=1/(2f_m)$, Nyquist Frequency: $2f_m$

サンプリング定理 (2)

- 帯域制限された場合のフーリエ変換対は次式で与えられる

$$X(f) = \int_0^{\infty} x(t)\exp(-j2\pi ft)dt \qquad (1)$$

$$x(t) = \int_{-f_m}^{f_m} X(f)\exp(j2\pi ft)df \qquad (2)$$

- *X(f)*をフーリエ級数展開すれば次式を得る

$$X(f) = \sum_{n=-\infty}^{\infty} C_n \cdot exp(j2\pi fn/(2f_m)) \qquad (3)$$

Sampling Theorem (2)

- Fourier Transform Pair with Limited Frequency Bandwidth

$$X(f) = \int_0^{\infty} x(t)\exp(-j2\pi ft)dt \qquad (1)$$

$$x(t) = \int_{-f_m}^{f_m} X(f)\exp(j2\pi ft)df \qquad (2)$$

- Fourier Series Expansion of $X(f)$ can be written as

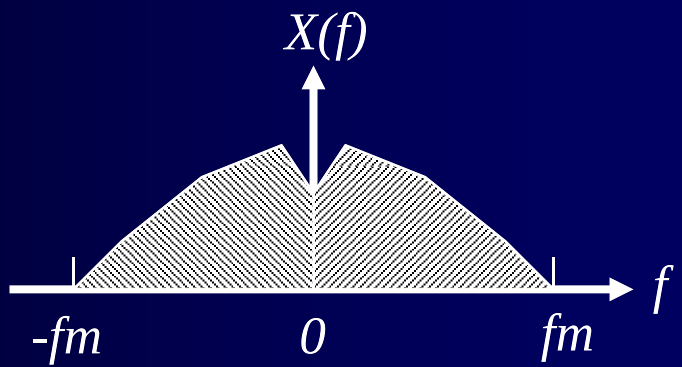
$$X(f) = \sum_{n=-\infty}^{\infty} C_n \cdot exp(j2\pi fn/(2f_m)) \qquad (3)$$

サンプリング定理 (3)

- ここで、フーリエ級数の係数 C_n は以下のようになる

$$
\begin{aligned}
C_n &= \frac{1}{2f_m}\int_{-f_m}^{f_m} X(f)\cdot\exp(-j2\pi fn/(2f_m))\,df \\
&= \frac{1}{2f_m}x\left(-\frac{n}{2f_m}\right)
\end{aligned}
\tag{4}
$$

- *X(f)*のスペクトル形状

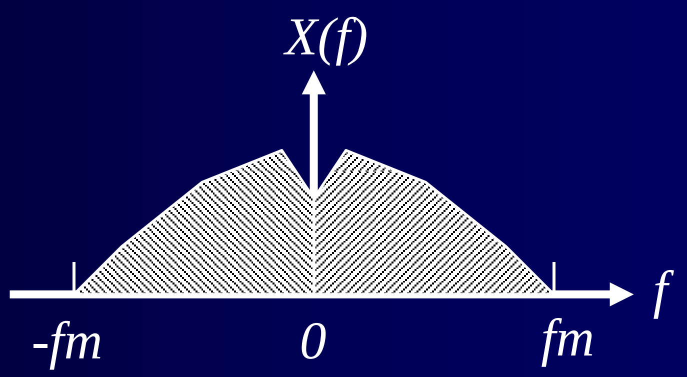


Sampling Theorem (3)

- Where C_n is written as follows

$$C_n = \frac{1}{2f_m}\int_{-f_m}^{f_m} X(f)\cdot \exp(-j2\pi fn/(2f_m))\,df$$
$$= \frac{1}{2f_m} x\left(-\frac{n}{2f_m}\right) \qquad (4)$$

- Shape of $X(f)$



サンプリング定理 (4)

- 式(3),(4)を式(2)に代入すると、次式を得る

$$x(t) = \int_{-f_m}^{f_m} \left[\sum_{n=-\infty}^{\infty} \frac{1}{2f_m} x\left(-\frac{n}{2f_m}\right) \cdot exp(j2\pi fn/(2f_m)) \right] \cdot \exp(j2\pi ft)\, df$$

$$\equiv \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) h_n(t)$$

- ここに、$h_n(t)$は

$$h_n(t) = \int_{-f_m}^{f_m} \frac{1}{2f_m} \exp\left(j2\pi f(t-\frac{n}{2f_m}) \right) df = \frac{\sin 2\pi f_m (t-\frac{n}{2f_m})}{2\pi f_m (t-\frac{n}{2f_m})}$$

Sampling Theorem (4)

- Substitute (3),(4) to (2), we obtain

$$x(t) = \int_{-f_m}^{f_m} \left[\sum_{n=-\infty}^{\infty} \frac{1}{2f_m} x\left(-\frac{n}{2f_m} \right) \cdot exp(j2\pi fn/(2f_m)) \right] \cdot \exp(j2\pi ft)\, df$$

$$\equiv \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m} \right) h_n(t)$$

- Where $h_n(t)$ is

$$h_n(t) = \int_{-f_m}^{f_m} \frac{1}{2f_m} \exp\left(j2\pi f(t - \frac{n}{2f_m}) \right) df = \frac{\sin 2\pi f_m (t - \frac{n}{2f_m})}{2\pi f_m (t - \frac{n}{2f_m})}$$

サンプリング定理 (5)

- 得られた結果

$$x(t)=\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)h_n(t)$$

- これはサンプリング間隔がDt = $1/(2f_m)$ に選ばれると、連続的な入力信号 $x(t)$ が離散的にサンプルされた信号（PAM信号）x_n と理想的な低域通過型フィルタのインパルス応答$h_n(t)$から再現できることを示している。

$$x(t)=\sum_{n=-\infty}^{\infty} x_n \frac{\sin 2\pi f_m(t-n\Delta t)}{2\pi f_m(t-n\Delta t)}$$

Sampling Theorem (5)

- The obtained result

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) h_n(t)$$

- The result shows that the continuous input $x(t)$ can be recovered from sampled sequence (PAM signal) x_n and the ideal low-pass filter's impulse response $h_n(t)$ if the sampling interval is chosen to Dt = $1/(2f_m)$

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \frac{\sin 2\pi f_m (t - n\Delta t)}{2\pi f_m (t - n\Delta t)}$$

例

- コンパクトディクスに記録されたオーディオ信号
  - １６ビットPCM
  - *.wav ファイル（PC/WS）
  - 再生周波数帯域: 0 – 22.05 kHz
  - サンプリング周波数: 44.1 kHz

Example

- Audio Recorded in Compact Disk
  - 16bit PCM
  - *.wav file in PC/WS
  - Playback Frequency Range: 0 - 22.05 kHz
  - Sampling Frequency: 44.1 kHz

数学の準備

- フーリエ級数展開
  - 関数 $x(t)$ の変数 t が区間 $[0, T]$ に限られていれば、$x(t)$ は次式のようにフーリエ級数（複素数を含んだ三角関数）に展開できる

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \exp(j2\pi nt/T)$$

  - ここに、C_n はフーリエ級数の係数である

$$C_n = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi nt/T)dt$$

Mathematical Preparation

- Fourier Series Expansion
  - If variable domain of function $x(t)$ is limited $[0, T]$, $x(t)$ can be expanded by Fourier series (complex trigonometric functions)

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \exp(j2\pi nt/T)$$

  - Here, C_n is Fourier Series Coefficients

$$C_n = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi nt/T)dt$$

復習:数列の和

- 数列の和の表現

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

Review: Summation of Series

- Representation for Summation of Series

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

復習：積分

- 定積分：積分範囲が決まっている積分

$$\int_1^3 x^2 dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$$

- 不定積分：積分範囲のない積分，微分の逆演算

$$\int x\,dx = \frac{1}{2}x^2$$

$$\int \sin x\,dx = -\cos x$$

Review: Integral

- Definite Integral : Integral Range is defined

$$\int_1^3 x^2 dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$$

- Indefinite Integral：Integral without Range, Anti-derivative

$$\int x\,dx = \frac{1}{2}x^2$$

$$\int \sin x\,dx = -\cos x$$

復習:複素数

- 複素数とは虚数 j を含む数

$$j=\sqrt{-1}$$

- 実数と虚数項を持つ複素数の加算および積の演算は

$$x=a+jb, \quad y=c+jd$$

とすると

$$\begin{aligned} x+y &= (a+c)+j(b+d) \\ xy &= (a+jb)(c+jd) \\ &= ac+j^2bd+j(ad+bc) \\ &= (ac-bd)+j(ad+bc) \end{aligned}$$

Review: Complex Number

- Complex Number has imaginary part j

$$j = \sqrt{-1}$$

- When we have two complex numbers,

$$x = a + jb, \quad y = c + jd$$

Addition and Multiplication of these numbers are

$$x + y = (a + c) + j(b + d)$$

$$\begin{aligned} xy &= (a + jb)(c + jd) \\ &= ac + j^2 bd + j(ad + bc) \\ &= (ac - bd) + j(ad + bc) \end{aligned}$$

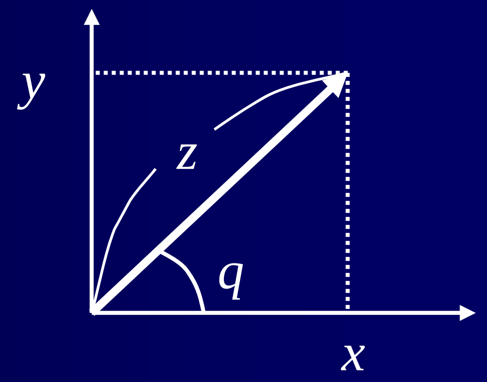
復習:三角関数

- sin, cos, tanは，どのように定義されるか？

$$\cos\theta = \frac{x}{z} \qquad (x = z\cos\theta)$$

$$\sin\theta = \frac{y}{z} \qquad (y = z\sin\theta)$$

$$\tan\theta = \frac{y}{x} \qquad (\tan\theta = \frac{\sin\theta}{\cos\theta})$$



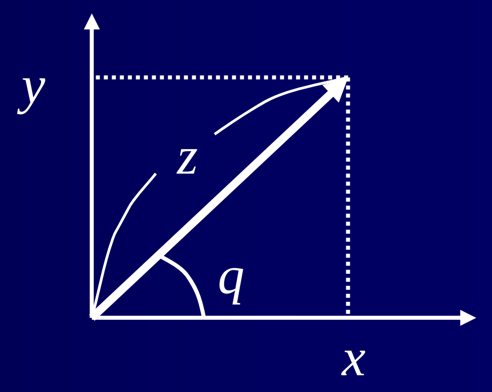
Review: Trigonometric Function

- How can sin, cos, tan be defined?

$$\cos\theta = \frac{x}{z} \qquad (x = z\cos\theta)$$

$$\sin\theta = \frac{y}{z} \qquad (y = z\sin\theta)$$

$$\tan\theta = \frac{y}{x} \qquad (\tan\theta = \frac{\sin\theta}{\cos\theta})$$



復習：指数関数

- 指数関数はどのように表現されるか？

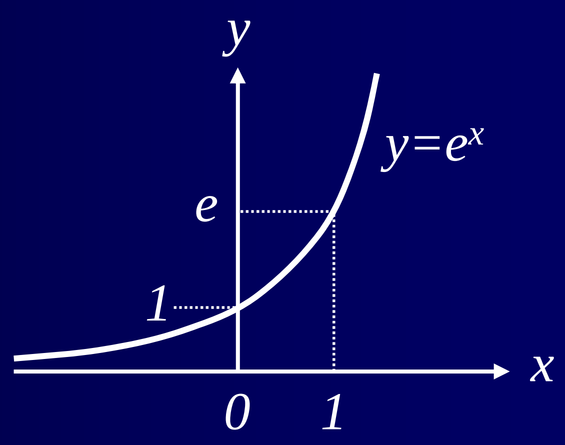
$$e^x = \exp(x)$$

- 指数関数の加算および積の演算は？

$$e^a e^b = e^{a+b}$$

$$e^a / e^b = e^{a-b}$$

$$e^{ab} = (e^a)^b$$



Review: Exponential Function

- How exponential function can be represented?

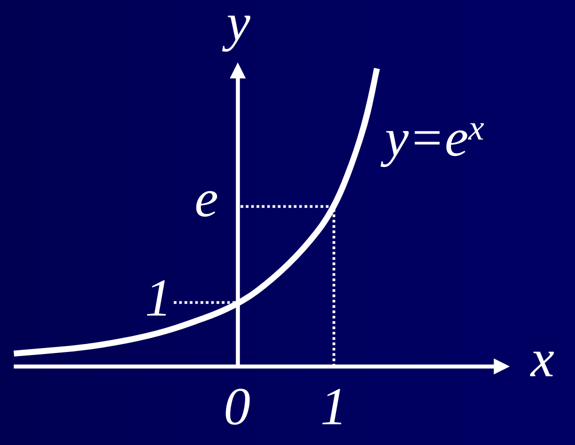
$$e^x = \exp(x)$$

- Addition and Multiplication of exponentials？

$$e^a e^b = e^{a+b}$$

$$e^a / e^b = e^{a-b}$$

$$e^{ab} = (e^a)^b$$



復習:指数関数 (2)

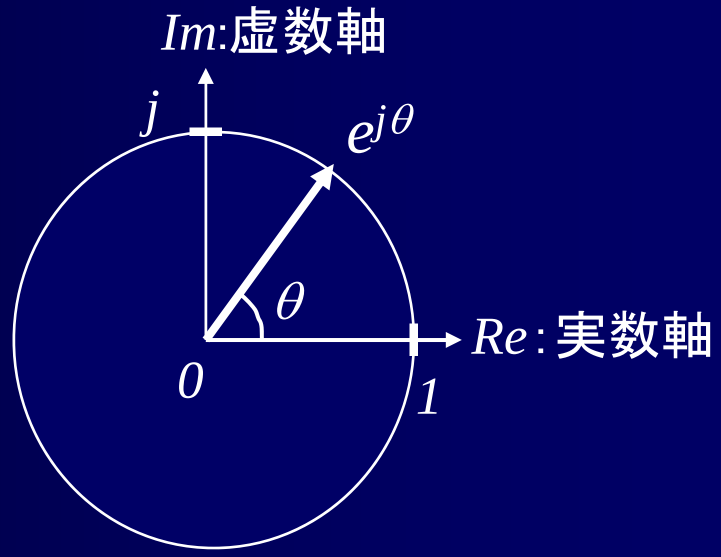
- 指数部に虚数項を持つとき，その数の意味は？

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- 複素指数関数の積は？

$$e^{ja}e^{jb} = e^{j(a+b)}$$

- 複素形式の利点は？
  - 波の表現が簡単
  - プログラムが容易



Review: Exponential Function (2)

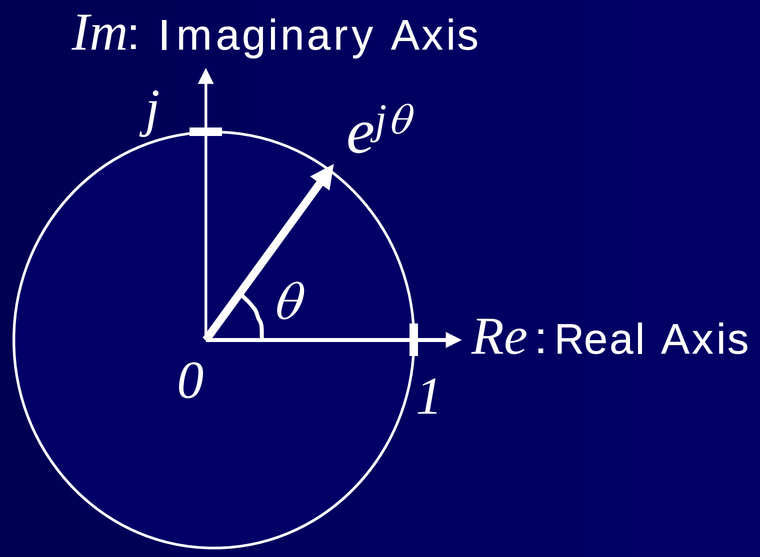
- What is the number having Imaginary number in exponential part?

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- Multiplication of these numbers

$$e^{ja}e^{jb} = e^{j(a+b)}$$

- Advantage of this expression
  - Simple wave notation
  - Easy programming



例

- $j^2=-1$ であることを，$e^{j\theta}(\theta=\pi/2)$を用いて証明せよ

Example

- Prove $j^2=-1$ by using $e^{j\theta}(\theta=\pi/2)$

数学の準備 (2)

- フーリエ級数関数は直交関数である
- 周期関数は直交関数で展開できる（なぜなら基本区間のパワーが有限であるから）
  - 周期関数　$x(t)=x(t+nT)$
  - 直交関数：内積が零
  - 関数 $x_1(t)$ と $x_2(t)$ が次の関係を持つとき

$$\frac{1}{T}\int_{-T/2}^{T/2} x_1(t)x_2{}^{*}(t)dt = 0$$

＊：複素共役

$x_1(t)$ と $x_2(t)$ は領域 $[-T/2, T/2]$ において直交する

Mathematical Preparation (2)

- Fourier Series Function is Orthogonal Function
- Periodic function can be expanded by Orthogonal Function since the power of basic domain is limited
  - Periodic function $x(t)=x(t+nT)$
  - Orthogonal Function: Inner Product equals zero
  - When function $x_1(t)$ and $x_2(t)$ has the next relation

$$\frac{1}{T}\int_{-T/2}^{T/2} x_1(t)x_2{}^*(t)dt = 0$$

*:Conjugate

$x_1(t)$ and $x_2(t)$ are orthogonal in domain $[-T/2, T/2]$

数学の準備 (3)

- ベクトルの直交と内積
  - 直交ベクトル $\{v_1,v_2\}$ を直交基底と呼ぶ
  - $\{v_1, v_2\}$ は、大きさが1のとき、正規直交系であると呼ぶ
  - 任意の入力ベクトル y は v_1 と v_2の線形結合で表される

$$\vec{y} = c_1\vec{v_1} + c_2\vec{v_2}$$

$$c_1 = (\vec{y} \cdot \vec{v_1})$$

$$c_2 = (\vec{y} \cdot \vec{v_2})$$

Mathematical Preparation (3)

- Orthogonal and Inner Product at Vector
  - Orthogonal Vectors $\{v_1, v_2\}$ are called orthogonal basis
  - $\{v_1, v_2\}$ are called orthonormal when they are unit vector (size 1)
  - Arbitrary input vector y can be described by a linear combination of v_1 and v_2

$$\vec{y} = c_1 \vec{v_1} + c_2 \vec{v_2}$$

$$c_1 = (\vec{y} \cdot \vec{v_1})$$

$$c_2 = (\vec{y} \cdot \vec{v_2})$$

例

- v_1 と v_2 が以下のような2次元ベクトルであるとき、直交していることを示せ

$$v_1 = \left\{ \frac{\sqrt{5}}{3}, \frac{2}{3} \right\}, \quad v_2 = \left\{ -\frac{2}{3}, \frac{\sqrt{5}}{3} \right\}$$

- 次のベクトル f を v_1 と v_2で展開せよ

$$f = \left\{ \frac{2\sqrt{5}}{3}, -\frac{1}{3} \right\}$$

Example

- Show the orthogonal characteristic when v_1 and v_2 are the following

$$v_1 = \left\{ \frac{\sqrt{5}}{3}, \frac{2}{3} \right\}, \quad v_2 = \left\{ -\frac{2}{3}, \frac{\sqrt{5}}{3} \right\}$$

- Expand f by v_1 and v_2

$$f = \left\{ \frac{2\sqrt{5}}{3}, -\frac{1}{3} \right\}$$

数学の準備 (4)

- n-次元ベクトルへの拡張

$$
\begin{aligned}
f &= c_1 \overrightarrow{v_1} + c_2 \overrightarrow{v_2} + \cdots + c_n \overrightarrow{v_n} \\
&= \sum_{i=1}^{n} c_i \overrightarrow{v_i}
\end{aligned}
$$

ここに

$$c_i = (\overrightarrow{f} \cdot \overrightarrow{v_i})$$

Mathematical Preparation (4)

- Extension to n-dimensional vector

$$\vec{f} = c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_n \vec{v_n} = \sum_{i=1}^{n} c_i \vec{v_i}$$

where

$$c_i = (\vec{f} \cdot \vec{v_i})$$

数学の準備 (5)

- 三角関数は領域 [-p, p]で正規直交系をなす

$$\{1, \sqrt{2}\cos x, \sqrt{2}\sin x, \sqrt{2}\cos 2x, \sqrt{2}\sin 2x, \cdots\}$$

なぜなら

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos nx \cos nx\, dx = 1 \qquad (n = 1,2,\cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin nx \sin nx\, dx = 1 \qquad (n = 1,2,\cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2}\cos nx\, dx = 0 \qquad (n = 1,2,\cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2}\sin nx\, dx = 0 \qquad (n = 1,2,\cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos mx \sin nx\, dx = 0 \qquad (n = 1,2,\cdots)(m = 1,2,\cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos mx \cos nx\, dx = 0 \qquad (n = 1,2,\cdots)(m = 1,2,\cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin mx \sin nx\, dx = 0 \qquad (n = 1,2,\cdots)(m = 1,2,\cdots)$$

Mathematical Preparation (5)

- Trigonometric functions are orthonormal at domain [-p, p]

$$\{1, \sqrt{2}\cos x, \sqrt{2}\sin x, \sqrt{2}\cos 2x, \sqrt{2}\sin 2x, \cdots\}$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos nx \cos nx \, dx = 1 \qquad (n = 1, 2, \cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin nx \sin nx \, dx = 1 \qquad (n = 1, 2, \cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2}\cos nx \, dx = 0 \qquad (n = 1, 2, \cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2}\sin nx \, dx = 0 \qquad (n = 1, 2, \cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos mx \sin nx \, dx = 0 \qquad (n = 1, 2, \cdots)(m = 1, 2, \cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos mx \cos nx \, dx = 0 \qquad (n = 1, 2, \cdots)(m = 1, 2, \cdots)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin mx \sin nx \, dx = 0 \qquad (n = 1, 2, \cdots)(m = 1, 2, \cdots)$$

数学の準備（6）

- 周期Tなる関数f(t)の実フーリエ級数展開は次式で与えられる

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (\omega_0 = 2\pi / T)$$

ここに

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \qquad (\omega_0 = 2\pi / T,\ n = 1,2,\cdots)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \qquad (\omega_0 = 2\pi / T,\ n = 1,2,\cdots)$$

Mathematical Preparation (6)

- Real Fourier Series Expansion of f(t) with period T

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (\omega_0 = 2\pi / T)$$

where

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \qquad (\omega_0 = 2\pi / T,\ n = 1,2,\cdots)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \qquad (\omega_0 = 2\pi / T,\ n = 1,2,\cdots)$$

数学の準備（7）

- 周期Tなる関数f(t)の複素フーリエ級数展開は次式で与えられる

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_0 t)$$

ここに

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(-jn\omega_0 t)\,dt$$

Mathematical Preparation (7)

- Complex Fourier Series Expansion of f(t) with period T

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_0 t)$$

where

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(-jn\omega_0 t)\,dt$$