

マルチメディア配信システム

- No.2 折返し歪とフィルタリング -

渡辺 裕

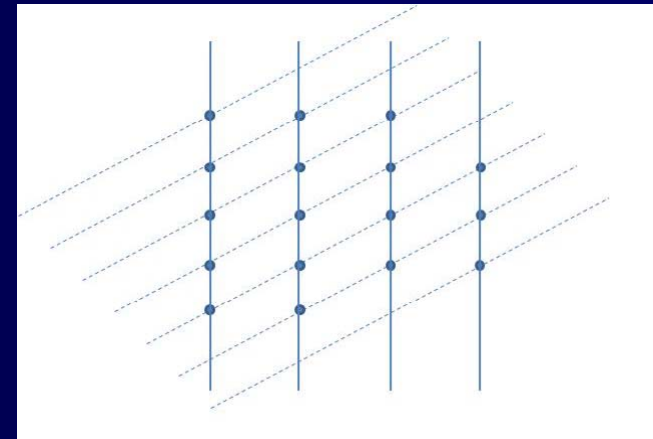
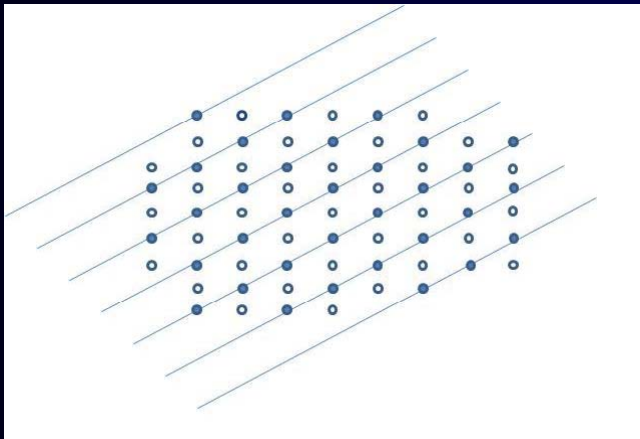
Multimedia Distribution System

- No.2 Aliasing Noise and Filtering -

Hiroshi Watanabe

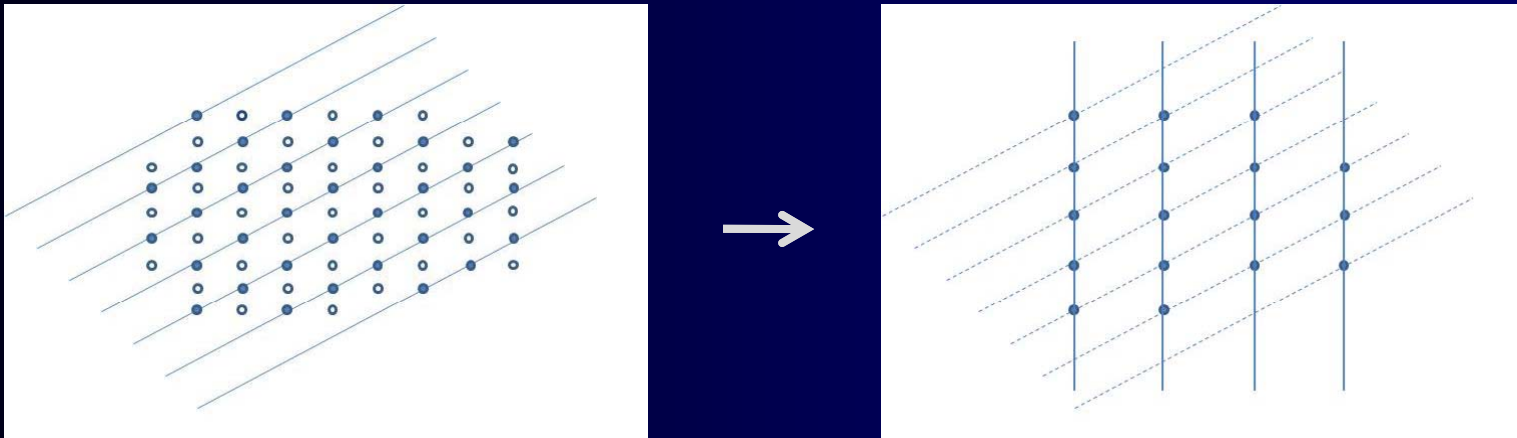
折返し歪 (1)

- サブサンプルによる空間領域での折り返し歪の例



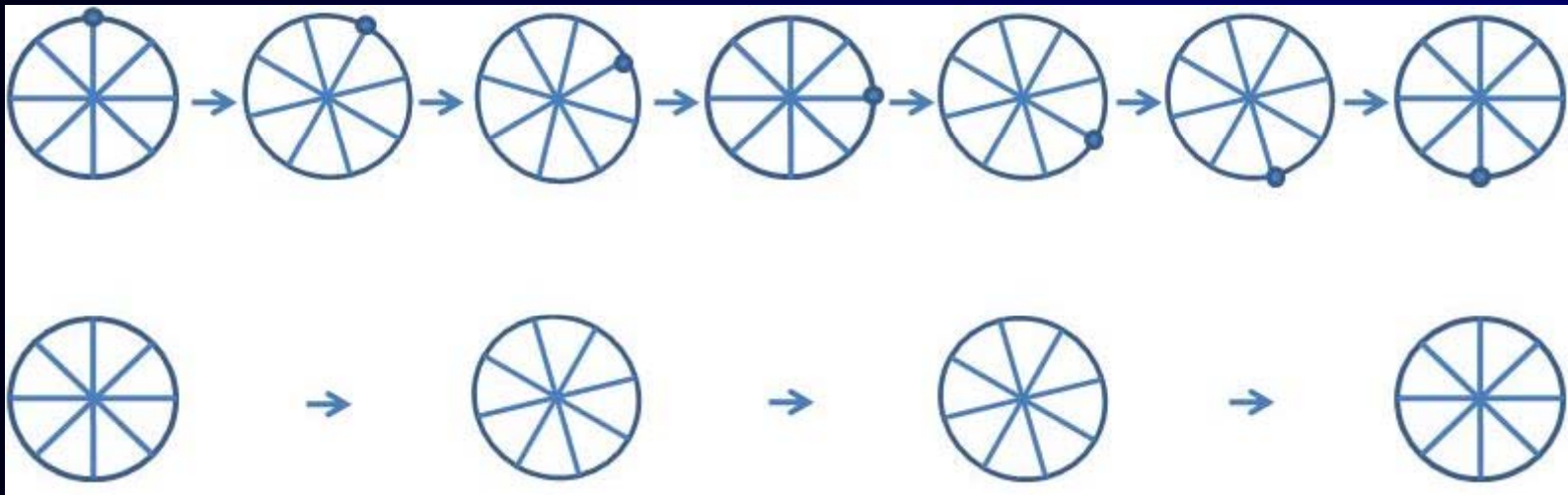
Aliasing Noise (1)

- Example of aliasing noise in a space domain by sub-sampling



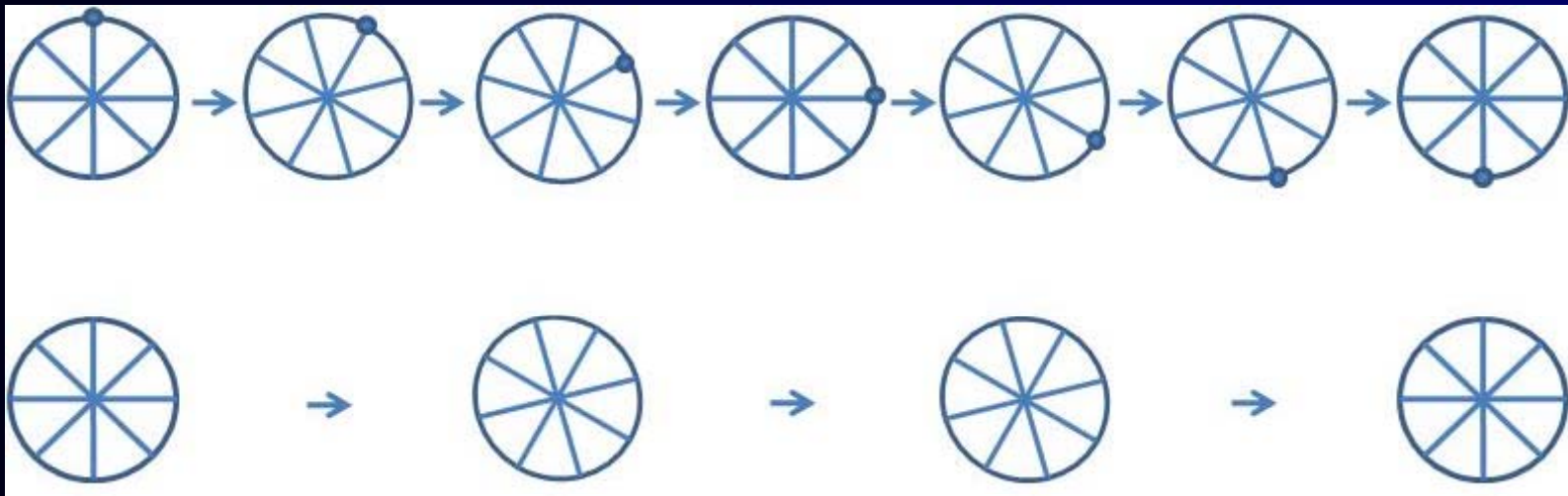
折返し歪 (2)

- フレーム間引きによる時間領域での折り返し歪の例



Aliasing Noise (2)

- Example of aliasing noise in a time domain by frame rate reduction



解像度変換とデジタルフィルタ

- N:1のサブサンプリングの際には, 通過帯域1/Nの低域通過型フィルタ (Low Pass Filter; LPF) を適用
- 伝達関数のz変換表現

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

- 例

$$H(z) = \frac{1}{4}z^1 + \frac{1}{2}z^0 + \frac{1}{4}z^{-1}$$

Resolution Conversion and Digital Filter

- For N:1 sub-sampling, Low Pass Filter (LPF) with pass-band 1/N
- Z-transform representation of transfer function

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

- Example

$$H(z) = \frac{1}{4}z^1 + \frac{1}{2}z^0 + \frac{1}{4}z^{-1}$$

画像処理 (1)

- Bitmap file
RGB 24 bit
720x576 pels



Image Processing (1)

- Bitmap file
RGB 24 bit
720x576 pels



画像処理 (2)

- 360x288 pels
without LPF
aliasing noise



Image Processing (2)

- 360x288 pels
without LPF
aliasing noise



画像処理 (3)

- 360x288 pels
with LPF
2D-(1,2,1)



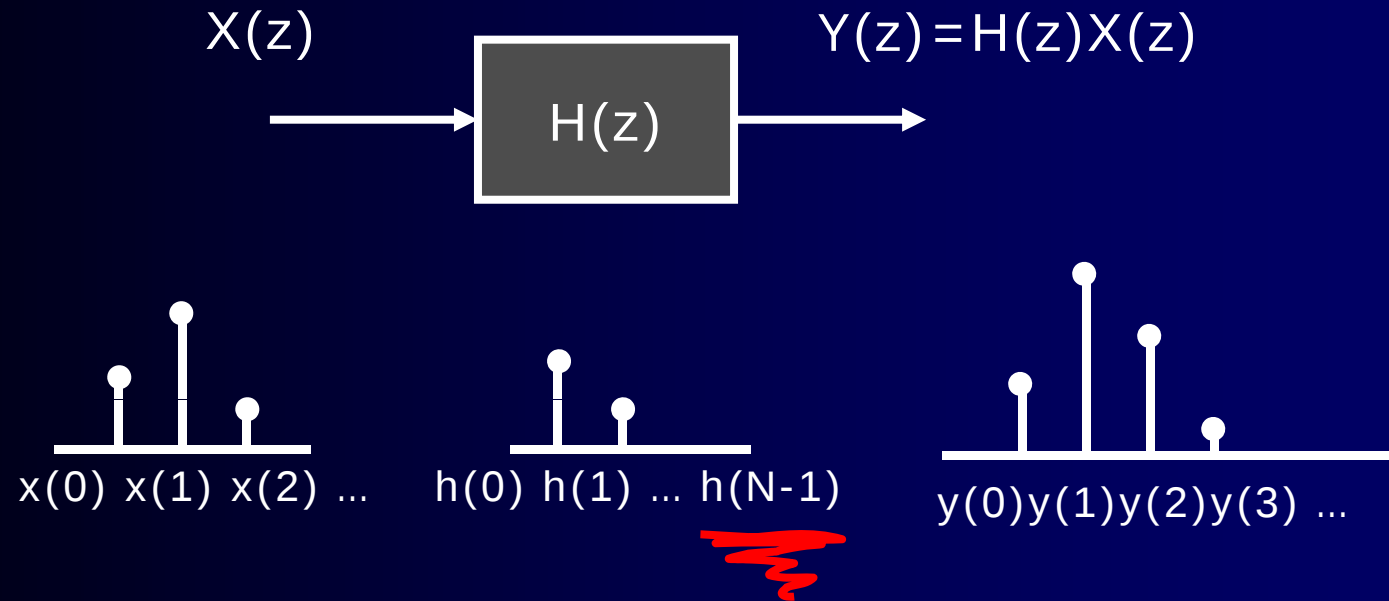
Image Processing (3)

- 360x288 pels
with LPF
2D-(1,2,1)



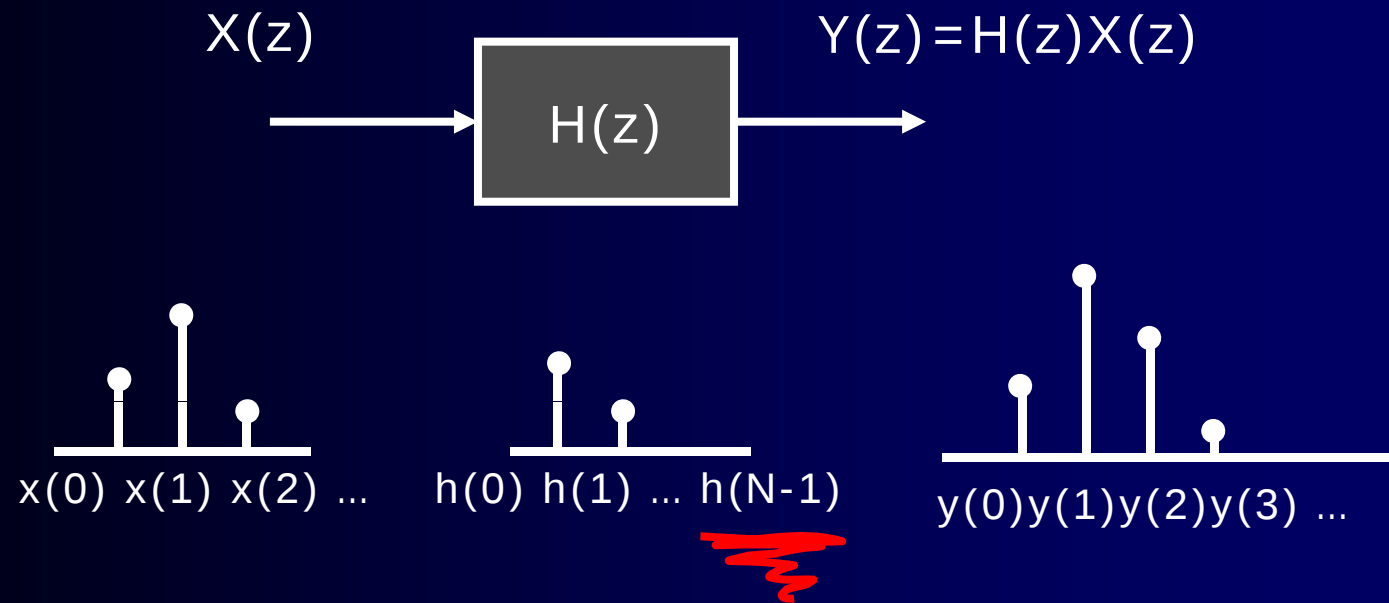
FIRフィルタ

■ z-変換によるシステム記述



FIR Filter

- System description by z-transform



係数の対応

- 入力信号

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$$

- インパルス応答

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2}$$

- 出力信号

$$Y(z) = H(z)X(z)$$

$$= (h_0 + h_1 z^{-1} + h_2 z^{-2})(a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3})$$

$$= h_0 a_0 + (h_0 a_1 + h_1 a_0) z^{-1} + (h_0 a_2 + h_1 a_1 + h_2 a_0) z^{-2}$$

$$+ (h_0 a_3 + h_1 a_2 + h_2 a_1) z^{-3} + (h_1 a_3 + h_2 a_2) z^{-4} + h_2 a_3 z^{-5}$$

Coefficients

- Input

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$$

- Impulse Response

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2}$$

- Output

$$Y(z) = H(z)X(z)$$

$$= (h_0 + h_1 z^{-1} + h_2 z^{-2})(a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3})$$

$$= h_0 a_0 + (h_0 a_1 + h_1 a_0) z^{-1} + (h_0 a_2 + h_1 a_1 + h_2 a_0) z^{-2}$$

$$+ (h_0 a_3 + h_1 a_2 + h_2 a_1) z^{-3} + (h_1 a_3 + h_2 a_2) z^{-4} + h_2 a_3 z^{-5}$$

係数の対応(2)

■ 係数の関係

$$y_0 = h_0 a_0$$

$$y_1 = h_0 a_1 + h_1 a_0$$

$$y_2 = h_0 a_2 + h_1 a_1 + h_2 a_0$$

$$y_3 = h_0 a_3 + h_1 a_2 + h_2 a_1$$

$$y_4 = h_1 a_3 + h_2 a_2$$

$$y_5 = h_2 a_3$$

Coefficients(2)

■ Relation of Coefficients

$$y_0 = h_0 a_0$$

$$y_1 = h_0 a_1 + h_1 a_0$$

$$y_2 = h_0 a_2 + h_1 a_1 + h_2 a_0$$

$$y_3 = h_0 a_3 + h_1 a_2 + h_2 a_1$$

$$y_4 = h_1 a_3 + h_2 a_2$$

$$y_5 = h_2 a_3$$

LPF(ローパスフィルタ)の設計

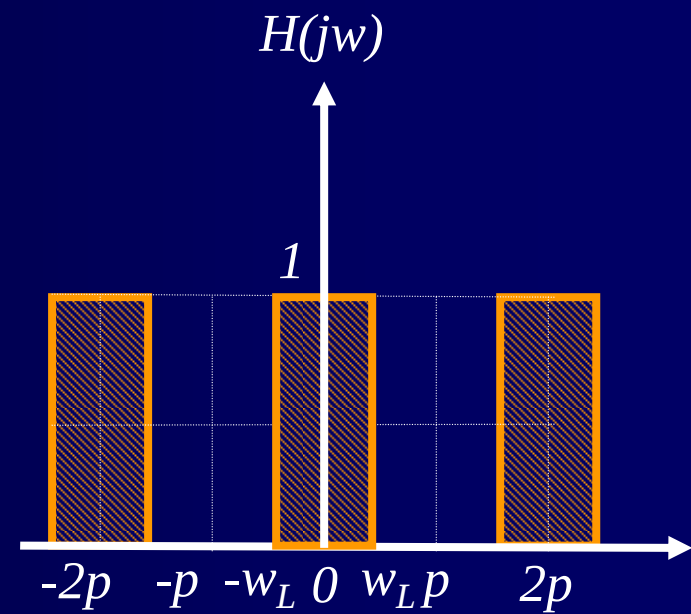
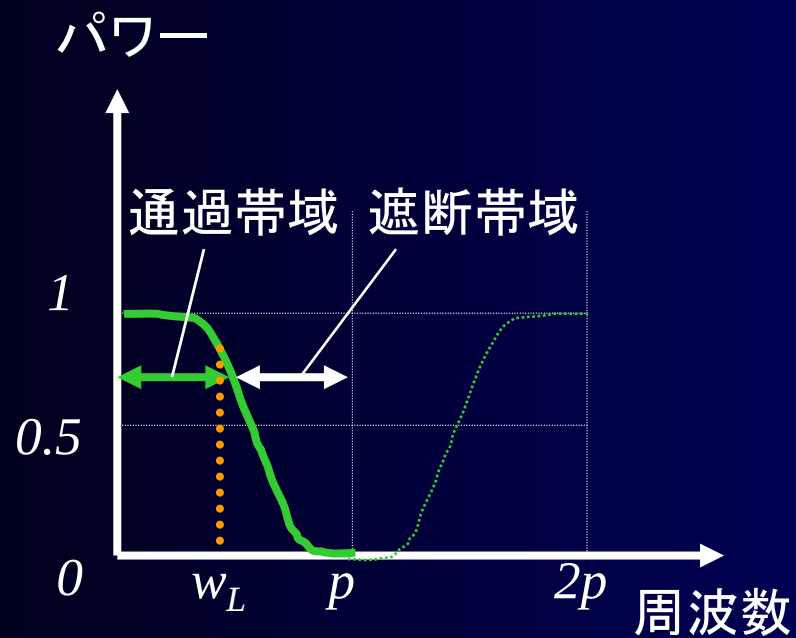
- FIRフィルタによるローパスフィルタ設計
 - ー 遮断周波数を決める
 - ー 伝達関数は理想ローパスフィルタを仮定する
 - ー フィルタ次数を決める
 - ー 係数を算出する
 - ー 係数と窓関数の積を求める

LPF Design

- Low-pass filter design by FIR filter
 - determine cut-off frequency
 - presume ideal low-pass filter as a transfer function
 - determine filter order
 - derive coefficients
 - multiply value of window function to coefficients

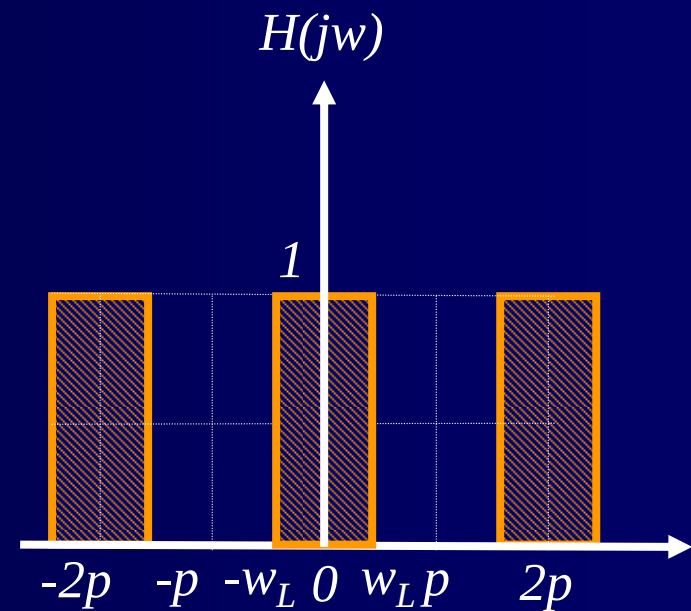
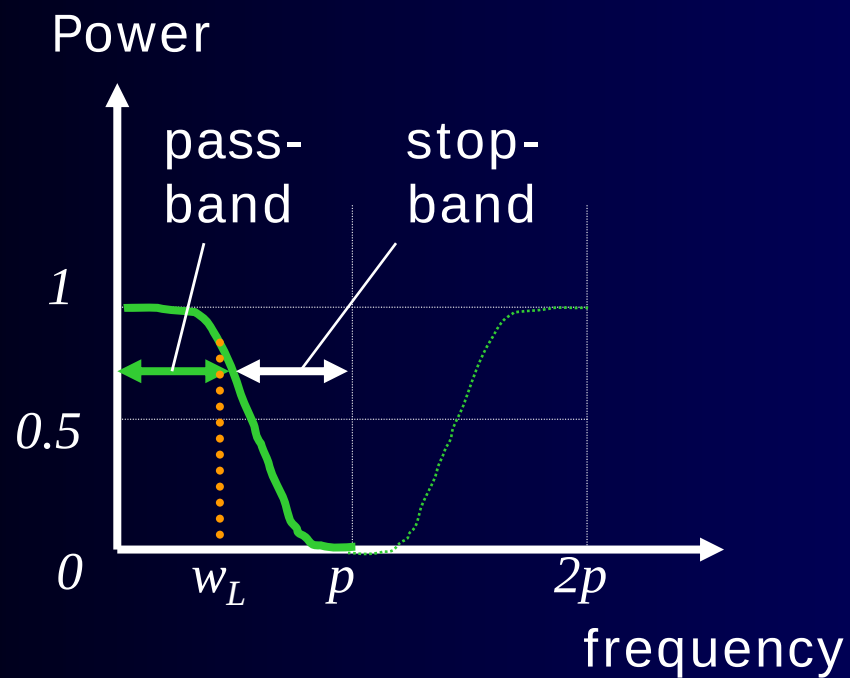
LPF遮断周波数

- 実際のローパスフィルタ特性と理想ローパスフィルタ特性
- 遮断周波数: w_L



LPF Cut-off Frequency

- Actual and ideal low-pass filter characteristics
- cut-off frequency: ω_L



LPF伝達関数と係数の関係

- 伝達関数が

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

のとき、係数 $h(k)$ は次式で計算できる

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \quad (k = 0, \dots, N-1)$$

Relation Between LPF Transfer Function and Coefficient

- When the transfer function is given by

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

coefficients $h(k)$ can be calculated by the following.

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \quad (k = 0, \dots, N-1)$$

LPF伝達関数と係数の関係(2)

■ なぜなら、

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{N-1} h(n) e^{-jn\omega} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \sum_{n=0}^{N-1} \int_{-\pi}^{\pi} h(n) e^{-j(n-k)\omega} d\omega \\ &= \frac{1}{2\pi} h(k) \int_{-\pi}^{\pi} d\omega \\ &= \frac{1}{2\pi} h(k) [\omega]_{-\pi}^{\pi} \\ &= h(k)\end{aligned}$$

Relation Between LPF Transfer Function and Coefficient(2)

■ Since,

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{N-1} h(n) e^{-jn\omega} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \sum_{n=0}^{N-1} \int_{-\pi}^{\pi} h(n) e^{-j(n-k)\omega} d\omega \\ &= \frac{1}{2\pi} h(k) \int_{-\pi}^{\pi} d\omega \\ &= \frac{1}{2\pi} h(k) [\omega]_{-\pi}^{\pi} \\ &= h(k)\end{aligned}$$

復習

- $[-p, p]$ における $\exp(jn\omega)$ の積分

$$\begin{aligned}\int_{-\pi}^{\pi} e^{jn\omega} d\omega &= \int_{-\pi}^{\pi} (\cos n\omega + j \sin n\omega) d\omega \\ &= [\sin n\omega - j \cos n\omega]_{-\pi}^{\pi} \\ &= (\sin n\pi - j \cos n\pi) - (\sin n(-\pi) - j \cos n(-\pi)) \\ &= 0\end{aligned}$$

Review

- Integral of $\exp(jn\omega)$ at $[p, p]$

$$\begin{aligned}\int_{-\pi}^{\pi} e^{jn\omega} d\omega &= \int_{-\pi}^{\pi} (\cos n\omega + j \sin n\omega) d\omega \\ &= [\sin n\omega - j \cos n\omega]_{-\pi}^{\pi} \\ &= (\sin n\pi - j \cos n\pi) - (\sin n(-\pi) - j \cos n(-\pi)) \\ &= 0\end{aligned}$$

例

- $H(z)=1/4z^0+1/2z^{-1}+1/4z^{-2}$ の場合の係数導出

$$h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) d\omega = \frac{1}{4}$$

$$h(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j\omega} d\omega = \frac{1}{2}$$

$$h(2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j2\omega} d\omega = \frac{1}{4}$$

Example

- Derivation of coefficients for $H(z)=1/4z^0+1/2z^{-1}+1/4z^{-2}$

$$h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) d\omega = \frac{1}{4}$$

$$h(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j\omega} d\omega = \frac{1}{2}$$

$$h(2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{4}e^0 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \right) e^{j2\omega} d\omega = \frac{1}{4}$$

係数の算出

- 遮断周波数 ω_L , フィルタの次数 N の場合

$$\begin{aligned} h(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} 1 d\omega \\ &= \frac{1}{2\pi} [\omega]_{-\omega_L}^{\omega_L} \\ &= \frac{\omega_L}{\pi} \end{aligned}$$

$$\begin{aligned} h(1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{j\omega}}{j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi} \end{aligned}$$

Derivation of Coefficients

- In case of cut-off frequency: ω_L , filter order N

$$\begin{aligned}h(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) d\omega \\&= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} 1 d\omega \\&= \frac{1}{2\pi} [\omega]_{-\omega_L}^{\omega_L} \\&= \frac{\omega_L}{\pi}\end{aligned}$$

$$\begin{aligned}h(1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{j\omega} d\omega \\&= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{j\omega} d\omega \\&= \frac{1}{2\pi} \left[\frac{e^{j\omega}}{j} \right]_{-\omega_L}^{\omega_L} \\&= \frac{\sin \omega_L}{\pi}\end{aligned}$$

係数の算出(2)

- 添字 k が負の場合

$$\begin{aligned} h(-1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{-j\omega}}{-j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi} \end{aligned}$$

Derivation of Coefficients(2)

- In case of negative sub-script k

$$\begin{aligned} h(-1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{-j\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{-j\omega}}{-j} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin \omega_L}{\pi} \end{aligned}$$

係数の算出(3)

- 一般にLPFの場合, 係数は次式で計算できる

$$h(0) = \frac{\omega_L}{\pi}$$

$$\begin{aligned} h(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{jk\omega}}{jk} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm(N-1)) \end{aligned}$$

Derivation of Coefficients(3)

- In general, coefficients for LPF is given by the following.

$$h(0) = \frac{\omega_L}{\pi}$$

$$\begin{aligned} h(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(j\omega) e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_L}^{\omega_L} e^{jk\omega} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{jk\omega}}{jk} \right]_{-\omega_L}^{\omega_L} \\ &= \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm(N-1)) \end{aligned}$$

LPFの設計

- 遮断周波数 $w_L = p/2$ を7次のフィルタで構成する場合

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

$$h(1) = h(-1) = \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin 3(\pi/2)}{3\pi} = \frac{-1}{3\pi} = -0.1061$$

LPF Design

- Cut-off frequency $\omega_L = p/2$, 7-th order filter

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

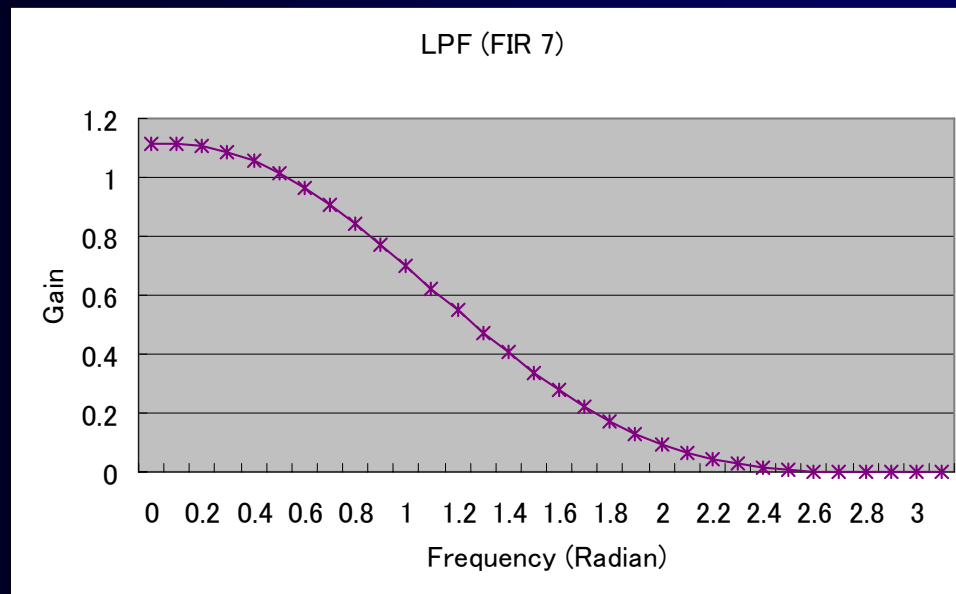
$$h(1) = h(-1) = \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin 3(\pi/2)}{3\pi} = \frac{-1}{3\pi} = -0.1061$$

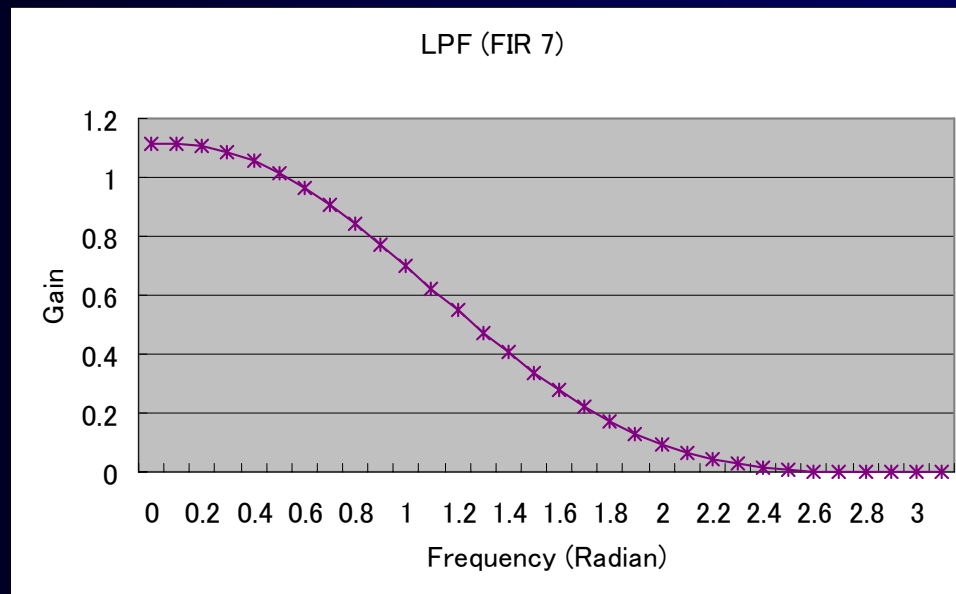
LPFの設計(2)

- 周波数特性
 - ー ゲインを正規化していないことに注意
 - ー 本来はY軸はdBで表示



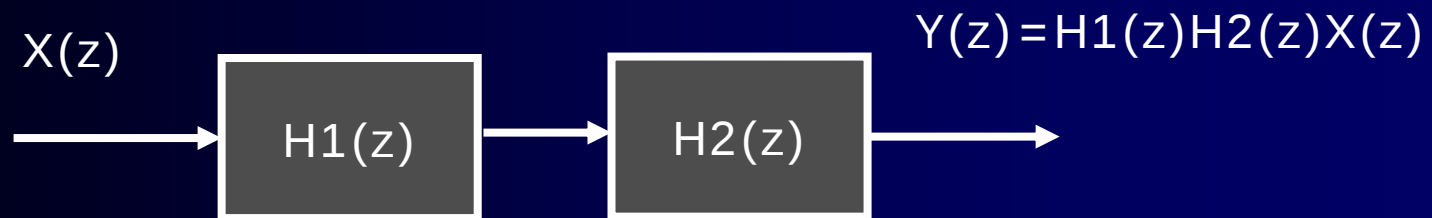
LPF Design(2)

- Frequency characteristics
 - Note that gain is not normalized
 - Gain is normally displayed by dB



遮断特性の改善

■ ローパスフィルタの多段接続

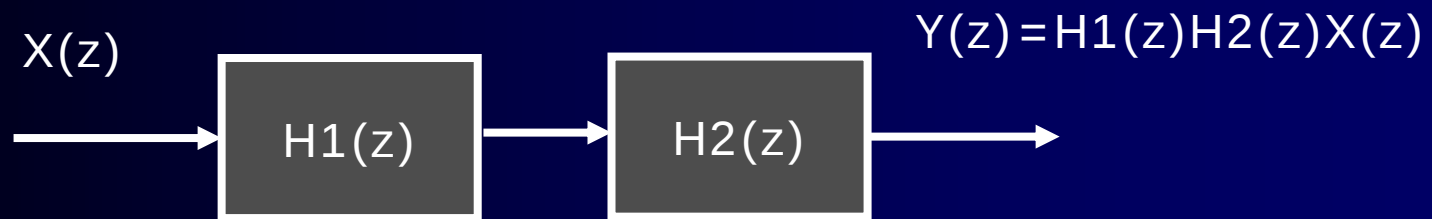


■ 窓関数

- 有限次数のフィルタ構成: ギブスの現象によりリップルを生じる
- 有限次数の打ち切りをなだらかに0に近づけるために, 窓関数を使用する

Cut-off Characteristics Improvement

- Multi-stage connection of LPF



- Window function
 - Limited order filter: occurs ripple by Gibbs phenomena
 - Use window function to make both ends coefficients gradually close to zero

窓関数

- ハミング窓

$$w(k) = 0.54 + 0.46 \cos\left(\frac{k\pi}{n}\right) \quad (-n \leq k \leq n)$$

- ハニング窓

$$w(k) = 0.5 \left(1 + \cos\left(\frac{k\pi}{n}\right) \right) \quad (-n \leq k \leq n)$$

Window Function

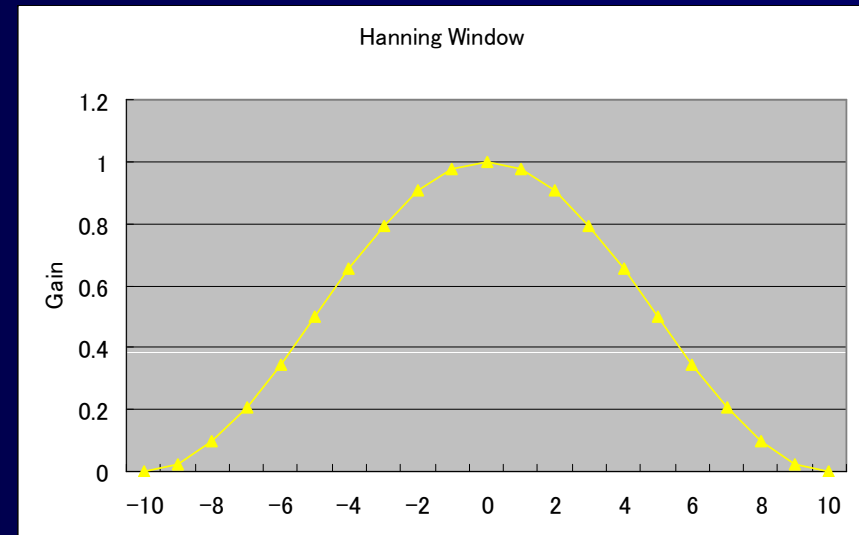
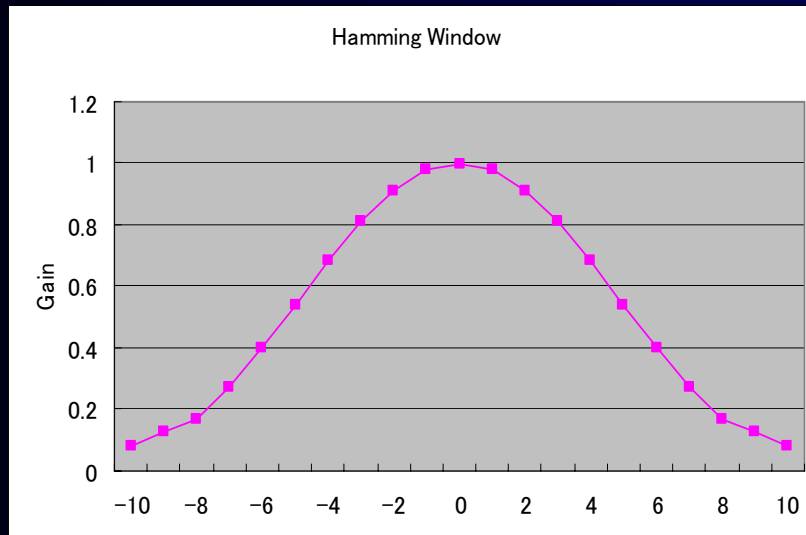
- Hamming Window

$$w(k) = 0.54 + 0.46 \cos\left(\frac{k\pi}{n}\right) \quad (-n \leq k \leq n)$$

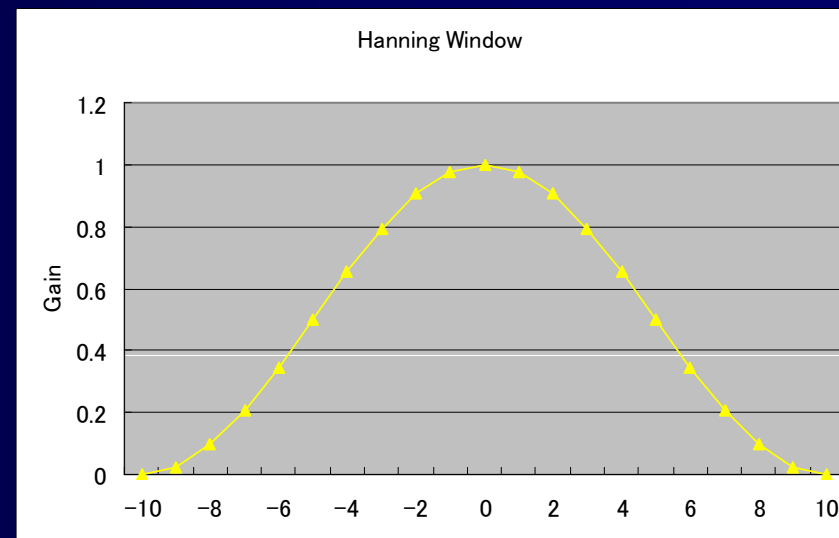
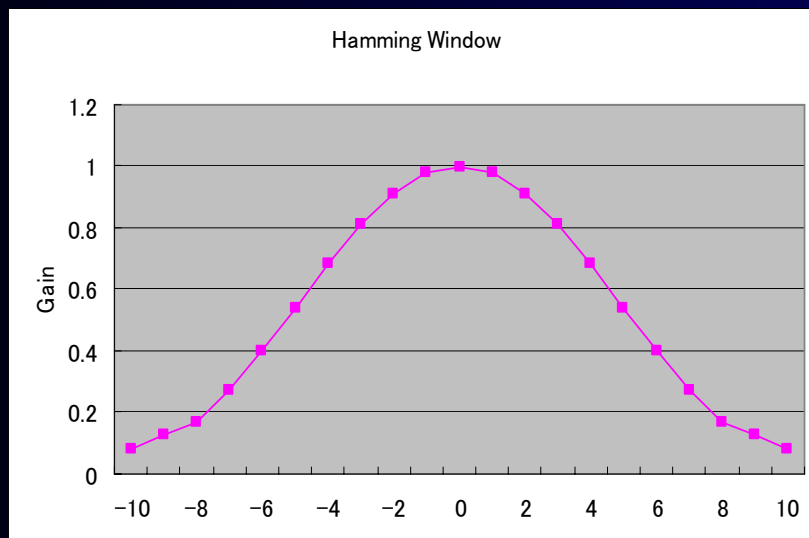
- Hanning Window

$$w(k) = 0.5 \left(1 + \cos\left(\frac{k\pi}{n}\right) \right) \quad (-n \leq k \leq n)$$

窓関数(2)



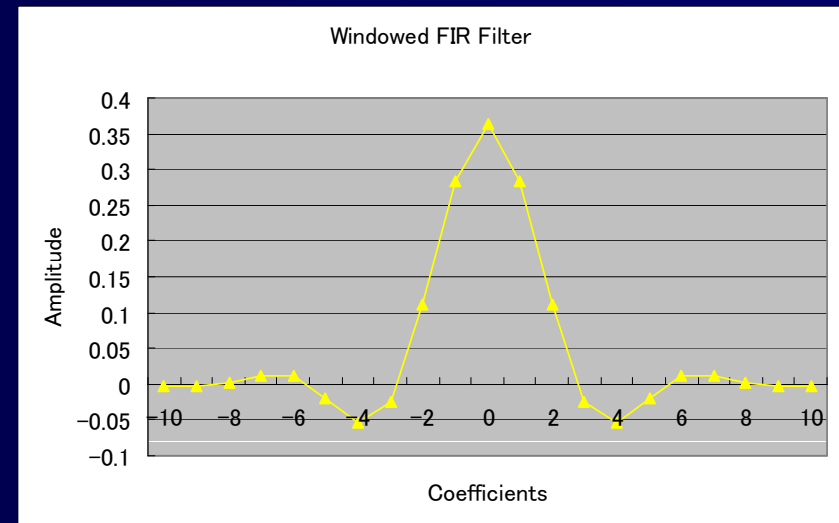
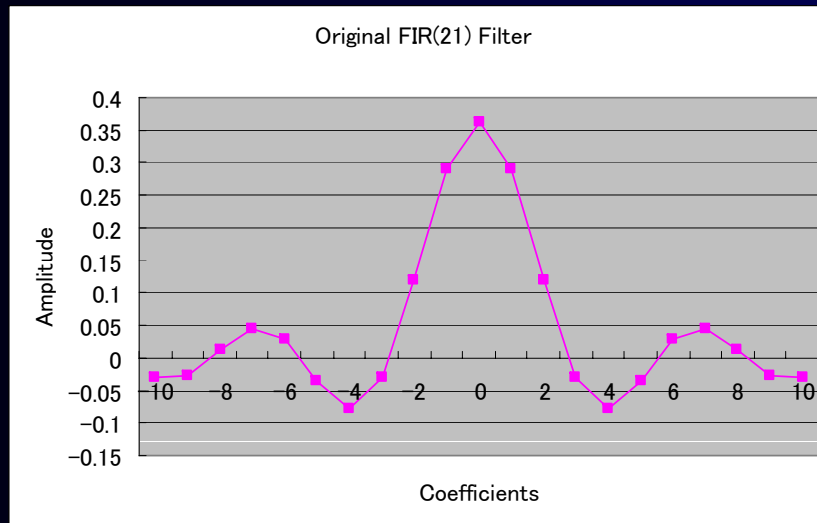
Window Function(2)



窓関数(3)

- フィルタ係数と窓関数値の積を最終的なフィルタ係数とする

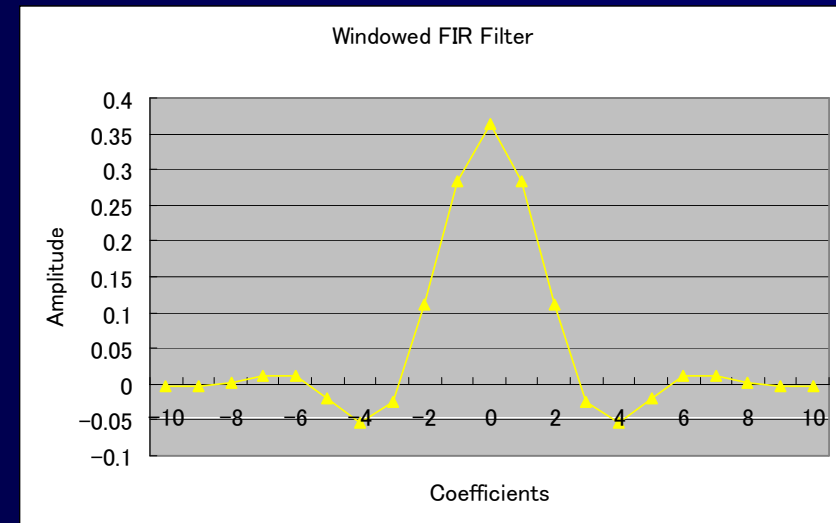
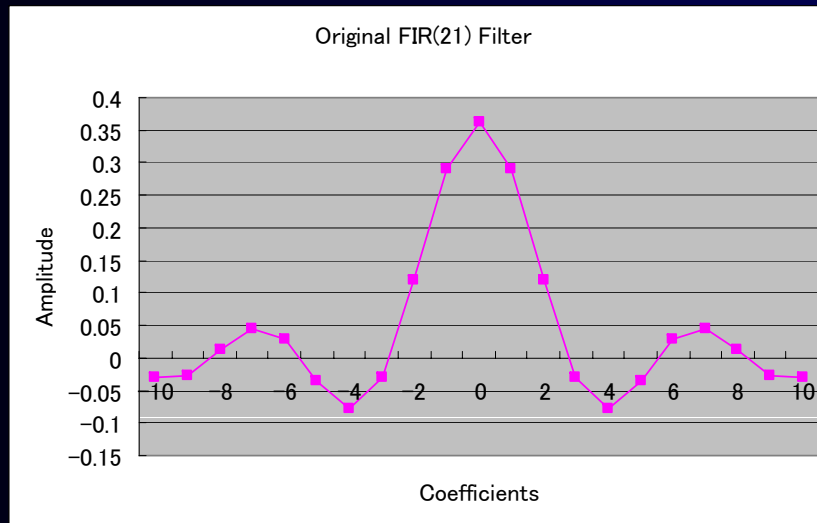
$$h_w(k) = h(k)w(k) \quad (-n \leq k \leq n)$$



Window Function(3)

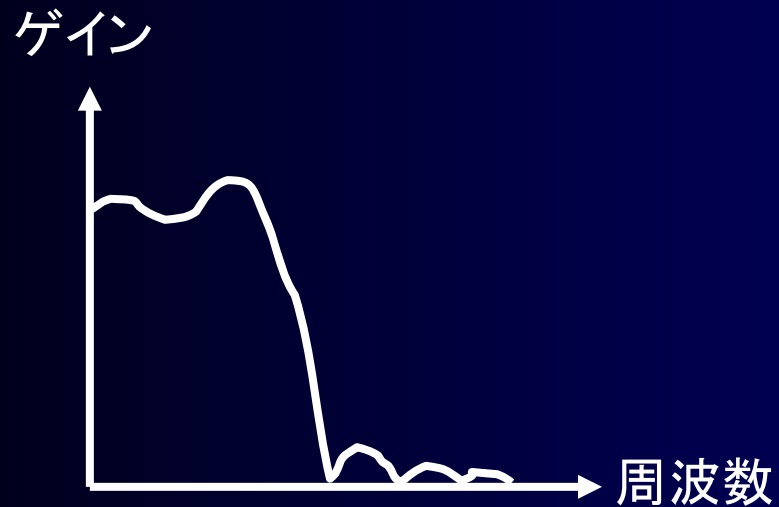
- Filter coefficients are multiplied by values of window function

$$h_w(k) = h(k)w(k) \quad (-n \leq k \leq n)$$

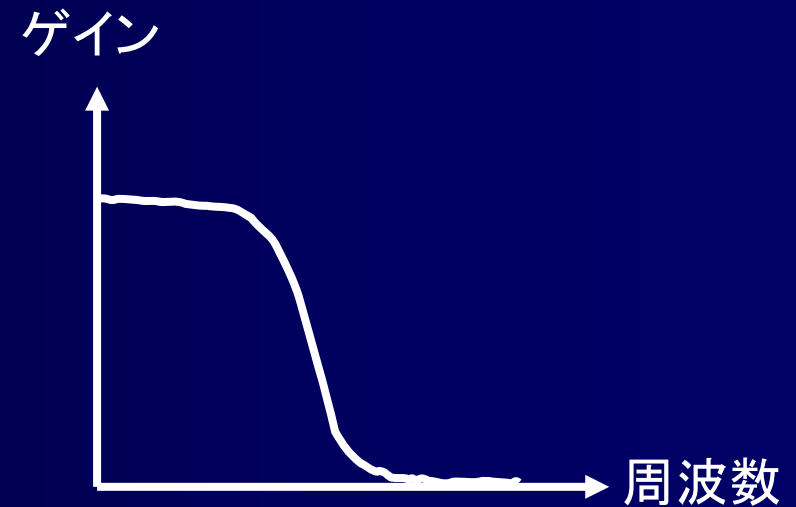


窓関数(4)

- 窓関数による周波数特性の変化



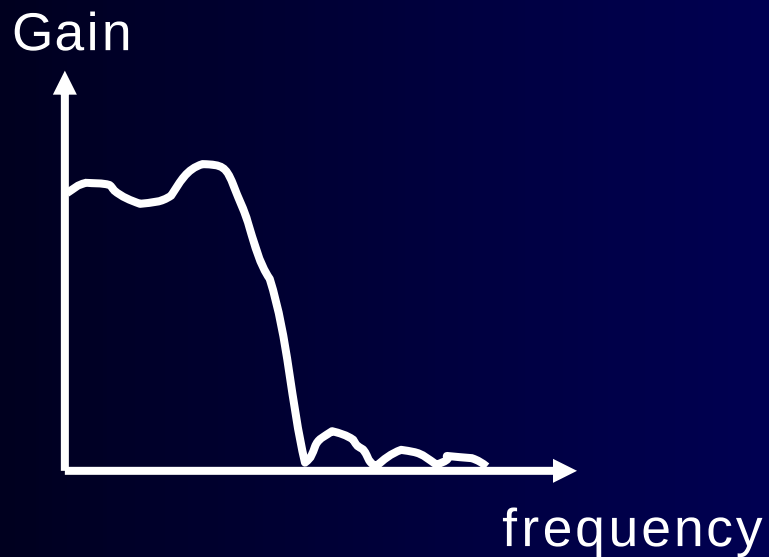
窓関数なし



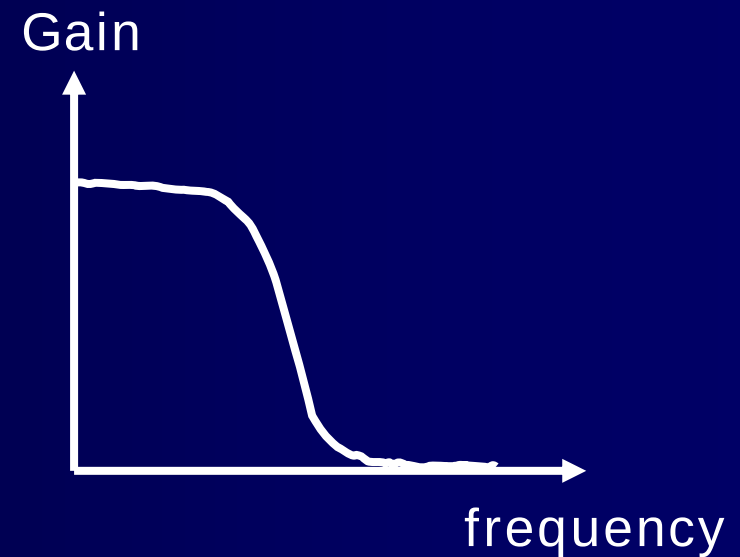
窓関数あり

Window Function(4)

- Change of frequency response by window function



without window
function



with window
function

HPF(ハイパスフィルタ)の設計

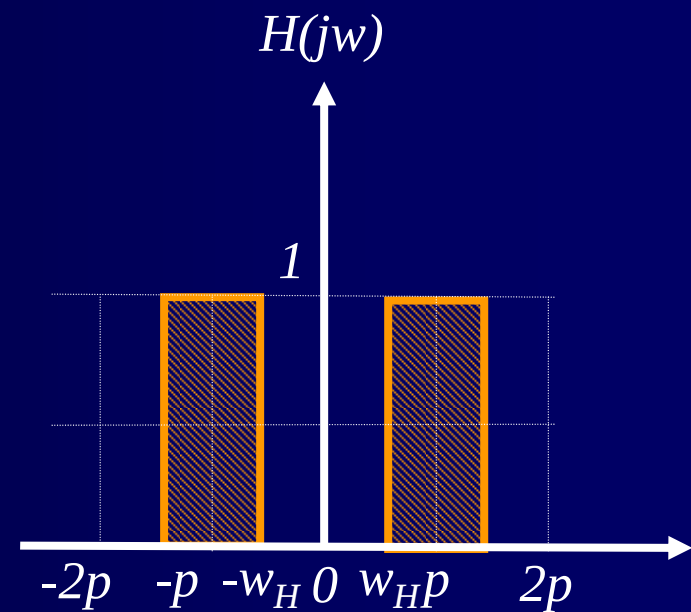
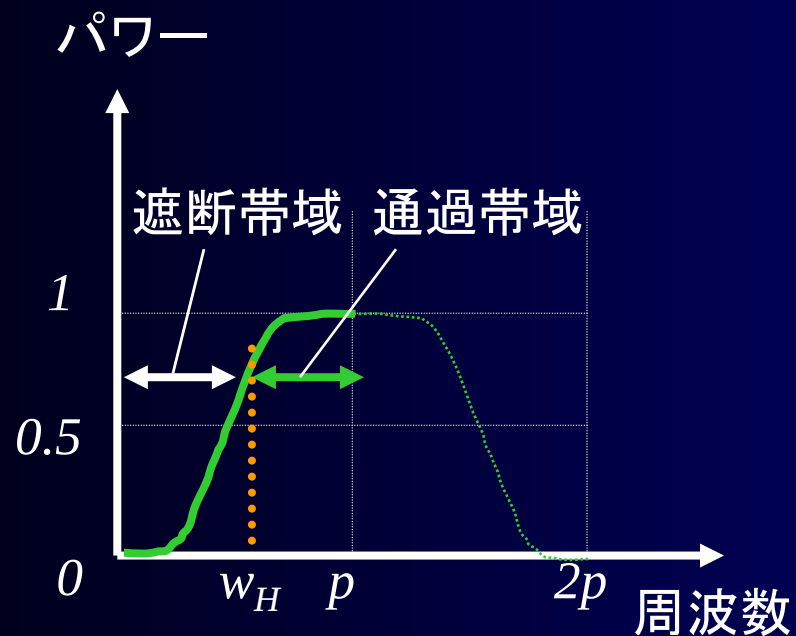
- FIRフィルタによるハイパスフィルタ設計
 - 遮断周波数を決める
 - 伝達関数は理想ハイパスフィルタを仮定する
 - フィルタ次数を決定する
 - ローパスフィルタの伝達関数をシフトして係数を算出する
 - 係数と窓関数の積を求める

HPF Design

- High-pass filter design by FIR filter
 - Determine cut-off frequency
 - presume ideal high-pass filter as a transfer function
 - determine filter order
 - derive coefficients by shifting low-pass filter's transfer function
 - multiply value of window function to coefficients

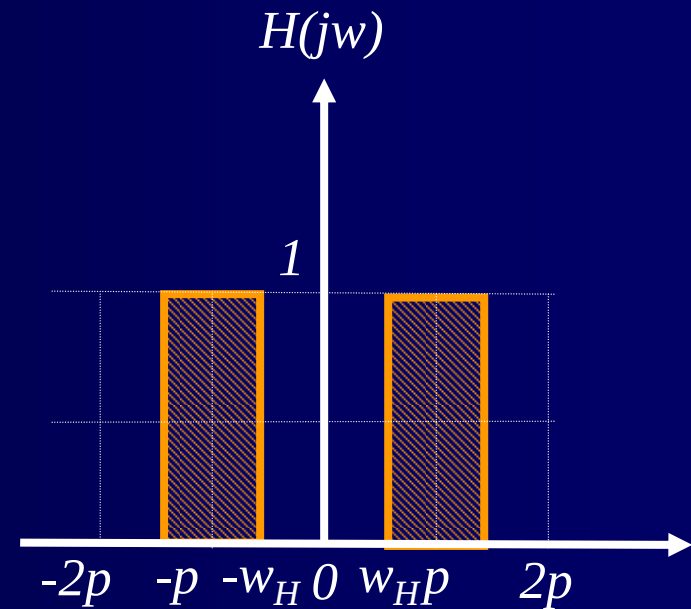
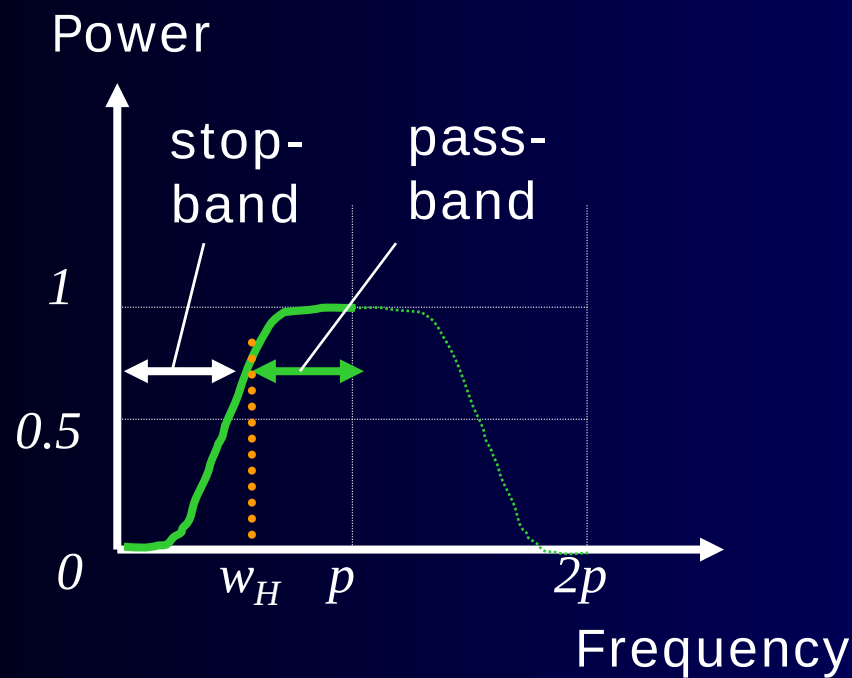
HPF遮断周波数

- 実際のハイパスフィルタ特性と理想ハイパスフィルタ特性
- 遮断周波数: $w_H = p - w_L$



HPF Cut-off Frequency

- Actual and ideal high-pass filter characteristics
- Cut-off frequency: $\omega_H = p - \omega_L$



HPF伝達関数と係数の関係

- ローパスフィルタの伝達関数が

$$H_L(z) = \sum_{k=-N}^N h(k)z^{-k}$$

のとき、ハイパスフィルタの伝達関数は、遮断周波数の関係

$$\omega_H = \pi - \omega_L$$

から $z = \exp(-jkw)$ において w を $w-p$ とおいたものに相当する

$$H_H(z) = \sum_{k=-N}^N h(k)e^{-jk(\omega-\pi)} = \sum_{k=-N}^N h(k)e^{-jk\omega}e^{jk\pi}$$

Relation Between HPF Transfer Function and Coefficients

- When LPF transfer function is given by

$$H_L(z) = \sum_{k=-N}^N h(k)z^{-k}$$

From the relation of cut-off frequency

$$\omega_H = \pi - \omega_L$$

HPF transfer function corresponds to the LPF's one replacing ω by $\omega - \pi$ at $z = \exp(-jk\omega)$

$$H_H(z) = \sum_{k=-N}^N h(k)e^{-jk(\omega-\pi)} = \sum_{k=-N}^N h(k)e^{-jk\omega}e^{jk\pi}$$

HPF伝達関数と係数の関係(2)

- 右辺の最後の項は

$$e^{jk\pi} = \cos k\pi + j \sin k\pi$$

であるから, k が偶数のときに 1 , 奇数のときに -1 となるから

$$H_H(j\omega) = (-1)^k H_L(j\omega)$$

となり, ハイパスフィルタの係数も $(-1)^k$ が乗じられた形となる

$$h_L(k) = (-1)^k \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm N)$$

ただし

$$h_L(0) = \frac{\omega_L}{\pi}$$

Relation Between HPF and Coefficients(2)

- Since

$$e^{jk\pi} = \cos k\pi + j \sin k\pi$$

the last term of the equation equals to 1 for even k , -1 for odd k

$$H_H(j\omega) = (-1)^k H_L(j\omega)$$

Thus, HPF coefficients are also given by multiplied form of $(-1)^k$

$$h_L(k) = (-1)^k \frac{\sin(k\omega_L)}{k\pi} \quad (k = \pm 1, \pm 2, \dots, \pm N)$$

where

$$h_L(0) = \frac{\omega_L}{\pi}$$

HPFの設計

- 遮断周波数 $w_H=p - w_L=p/2$ を7次のフィルタで構成する場合

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

$$h(1) = h(-1) = -\frac{\sin(\pi/2)}{\pi} = -\frac{1}{\pi} = -0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$h(3) = h(-3) = -\frac{\sin 3(\pi/2)}{3\pi} = -\frac{-1}{3\pi} = 0.1061$$

HPF Design

- Cut-off frequency $w_H=p - w_L=p/2$, 7-th order filter

$$h(0) = \frac{\pi/2}{\pi} = 0.5$$

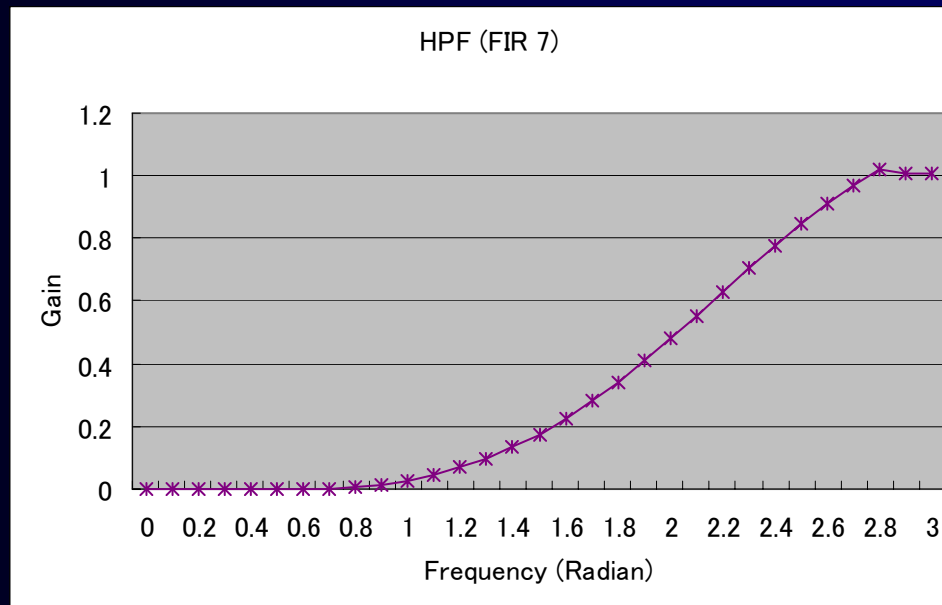
$$h(1) = h(-1) = -\frac{\sin(\pi/2)}{\pi} = -\frac{1}{\pi} = -0.3185$$

$$h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

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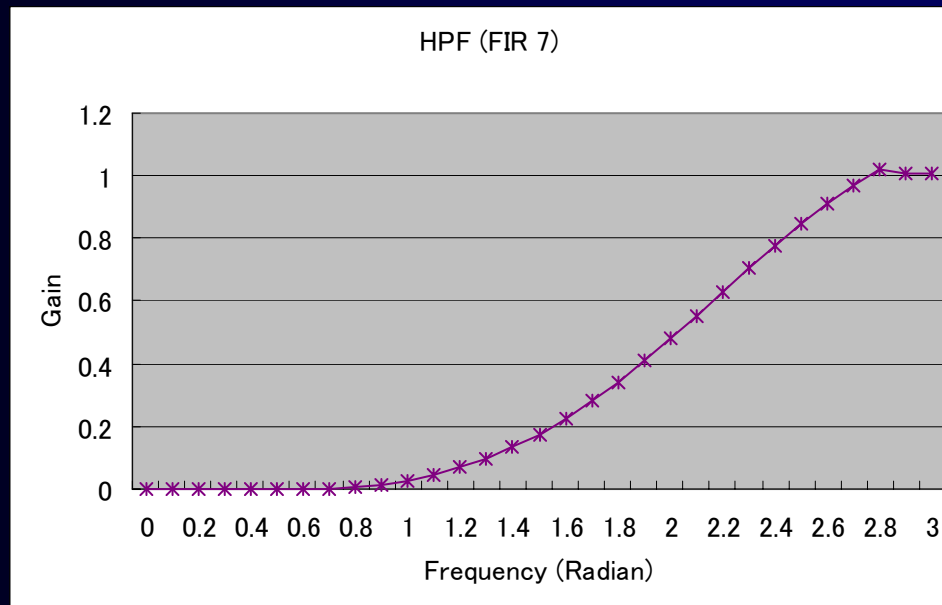
HPFの設計(2)

- 周波数特性
 - ー ゲインを正規化していないことに注意
 - ー 本来はY軸はdBで表示



HPF Design(2)

- Frequency characteristics
 - Note that gain is not normalized
 - Gain is normally displayed by dB



係数近似

- 係数の分数近似
- 分母が2のべき乗

$$h(0) = 0.5 \approx 8/16$$

$$h(1) = -0.3185 \approx -5/16$$

$$h(2) = 0$$

$$h(3) = 0.1061 \approx 1/16$$

Coefficient Approximation

- approximate coefficients by fractional number
- denominator is approximated by two's power

$$h(0) = 0.5 \approx 8/16$$

$$h(1) = -0.3185 \approx -5/16$$

$$h(2) = 0$$

$$h(3) = 0.1061 \approx 1/16$$

問題

- 遮断周波数 $\omega_L = 3p/4$, 3次FIR型のLPFを設計せよ

Quiz

- Design 3rd order FIR filter having cut-off frequency $w_L = 3p/4$